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THE NIKODYM THEOREMS FOR OPERATOR MEASURES

Charles Swartz

Two of the most interesting and useful results in measure theory are the Nikodym Boundedness Theorem ([9] IV. 9.8) and the Nikodym Convergence Theorem ([9] III. 7.4). In recent times these theorems have received a great deal of attention and have been generalized in several directions. For example, in [8], 8.6, the Nikodym Convergence Theorem has been generalized to the case of group-valued measures and in [10] the condition that the domain of the set functions is a σ-algebra has been relaxed. (See the discussion on pages 31–36 of [3].) When dealing with an integration theory that involves the integration of vector-valued functions with respect to operator-valued measures ([1], [2], [4]), it would be desirable to have versinos of the Nikodym Theorems that are appropriate for operator-valued measures. When dealing with vectorvalued measures the natural norm that is employed on the measures is the scalar semi-variation norm ([2], [3]) whereas when one deals with operator-valued measures the semi-variation norm ([3], [4]) is employed. For infinite dimensional spaces it is well-known that these norms are not equivalent ([12]). In this note we consider versions of the two Nihodym Theolems that seem appropriate for the case of operator-valued measures.

Throughout the paper let X, Y be (real) B-spaces with $L(X,Y)$ the space of bounded linear operators from X into Y equipped with the uniform norm. Let S be a non-void set and Σ a σ-algebra of subsets of S. If $\mu\colon \Sigma \to L(X,Y)$ is finitely additive, its semi-variation (with respect to $L(X,Y)$) is

$$\hat{\mu}(E) = \sup \left\| \sum_{i=1}^{n} \mu(E_i) x_i \right\|, \tag{1}$$

where the supremum is taken over all finite partitions $\{E_i\}$ of E and all $x_i \in X$ with $\|x_i\| \leq 1$ ([2] I. 4.1). If $\mu\colon \Sigma \to L(X,Y)$ is finitely additive, μ is said to be strongly bounded if $\hat{\mu}(E_j) \to 0$ whenever $\{E_j\}$ from Σ decreases to the empty set (Dobrakov ([4]) uses the term continuous.) A strongly bounded operator measure has finite semi-variation ([5] Th. 5 Cor.), and since $\|\mu(E)\| \leq \hat{\mu}(E)$ for $E \in \Sigma$, a strongly bounded operator measure is countably additive in the uniform operator topology. (The converse is false, [4), Example 7.) Let sca $(\Sigma, L(X,Y))$ = sca be the space of all strongly bounded set functions from Σ into $L(X,Y)$ and equip sca with its semi-variation norm.

One of the vector versions of the Nikodym Boundedness Theorem ([3] I. 3.I) is the following:

THEOREM 1. *Let $\mu_\alpha\colon \Sigma \to X$ $(\alpha \in I)$ be a family of bounded finitely additive measures. If*

$$\sup\{\|\mu_\alpha(E)\|\colon \alpha \in I\} < \infty \text{ for each } E \in \Sigma, \text{ then} \tag{2}$$

$$\sup\{\hat{\mu}_\alpha(S)\colon \alpha \in I\} < \infty, \tag{3}$$

where $\hat{\mu}$,is the scalar semi-variation of μ, (i.e., with respect to $X = L(R,X)$).

We now formulate the analogue of Theorem 1 for operator-valued measures belonging to sca. Let $\mathcal{S}(\Sigma,X)$ be the space all X-valued Σ-simple functions equipped with the uniform norm; if X is the scalar field, set $\mathcal{S}(\Sigma,X) = \mathcal{S}(\Sigma)$. A finitely additive set function $\mu\colon \Sigma \to (X,Y)$ with finite semi-variation induces a bounded linear operator, still denoted by μ, from $\mathcal{S}(\Sigma,X)$ into Y via $\mu(f) = \int_S f d\mu$ and $\|\mu\| = \hat{\mu}(S)$ ([4]). Theorem 1 can be rephrased to read, if the $\mu_\alpha\colon \mathcal{S}(\Sigma) \to X$ are pointwise bounded on $\mathcal{S}(\Sigma)$, then $\{\mu_\alpha\}$ is norm bounded. Thus, analogues of conditions (2) and (3) for operator measures would be

$$\{\mu_\alpha\} \text{ is pointwise bounded on } \mathcal{S}(\Sigma,X) \text{ and} \tag{2$'$}$$

$$\{\mu_\alpha\} \text{ is norm-bounded in } L(\mathcal{S}(\Sigma,X),Y). \tag{x'}$$

(From the Uniform Boundedness Principle $(2')$ is equivalent to $\sup_\alpha \|\mu_\alpha(E)\| < \infty$ for each E.) The following example shows that $(3')$ does not follow from $(2')$ for operator measures, i.e., the straight-forward analogue of Theorem 1 is not valid for measures in sca.

EXAMLE 2. Let $\{e_k\}_{k=1}^\infty$ be the canonical basis vectors in l^p, $e_k = \{\delta_{kj}\}_{j=1}^\infty$. Define $T_k\colon I^p \to l^p (1 \le p \le \infty)bxT_k x = x_k e_k$, where $x = \{x_j\} \in l^p$. Let $\mathcal{P}$ be the power set of the positive integers N and $\sigma \subseteq N$ and $n \in N$ set $\sigma(n) = \sigma \cap \{1, \dots, n\}$. Define $\mu_n\colon P \to L(l^p, l^p)$ by $\mu_n(\sigma) = \sum_{j \in \sigma(n)} T_j$. Since the series ΣT_j is subseries convergent in the strong operator topology the set $\{\|\mu_n(\sigma)\|\colon n\}$ is bouned for each $\sigma \subseteq N$. However, for each $n, \hat{\mu}_n(N) \ge \|\sum_{k=1}^n T_k e_k\| = n^{1/p}$ so that $\{\hat{\mu}_n(N)\colon n\}$ is unbounded.

We now show that a boundedness result similar to Theorem 1 can be obtained by replacing the space $\mathcal{S}(\Sigma,X)$ in $(2')$ bu a slightly larger space. Let $\mathcal{E}(\Sigma,X)$ be the space of all bounded, X-valued, Σ-elementary functions, i.e.; $f\colon S \to C$ of the form $f = \sum_{j=1}^\infty C_{Ej} x_j$, where the $\{E_j\} \subseteq \Sigma$ are disjoint and the $\{x_j\} \subseteq X$ are bounded.

(Here, C_E denotes the characteristic functions of the set E.) Recall $\mathcal{E}(\Sigma, X)$ is not generally complete with respect to the sup-norm but is dense in the space of all bounded X valued Σ-measurable functions, $\mathcal{B}(\Sigma, X)$ ([3], II, 1.2). If $\mu \in$ sca, μ, induces a bounded linear operator $\mu: \mathcal{E}(\Sigma, X) \to Y$ via $\mu(f) = \int_S f d\mu$ and $\|\mu\| = \hat{\mu}(S)$. ([4] Th. 5); the strong boundedness of μ is needed here to guarantee the integrability of each $f \in \mathcal{E}(\Sigma, X)$. By replacing the space $\mathcal{E}(\Sigma, X)$ of condition (2') by $\mathcal{E}(\Sigma, X)$, we obtain a version of the Nikodym Boundedness Theorem for sca.

THEOREM 3. *Let $\mu_\alpha \in$ sca, $\alpha \in I$. If*

$$(2'') \qquad \{\mu_\alpha : \alpha I\} \text{ is pointwise bounded on } \mathcal{E}(\Sigma, X),$$

then $(3)'$ *holds.*

PROOF: The semi-variation of any $\mu \in$ sca has the properties that $\hat{\mu}$ is countably subadditive ([2] I. 4.2.3), strongly bounded and $\hat{\mu}(A \cup B) + \hat{\mu}(B) \geq \hat{\mu}(A)$ for A, $B \in \Sigma$, $A \cap B = \emptyset$. Suppose $(3')$ does not hold. Then from the properties of $\hat{\mu}$ observed above end the proofs of either Theorem 4.4 of [7] or the Lemma of [6], there is disjoint sequence $\{E_j\} \subseteq \Sigma$ and a sequence $\{\mu_n\}$ such that $\sup \hat{\mu}(E_n) = \infty$. Then there is a subsequence such that $\hat{\mu}_{n_k}(E_{n_k}) \to \infty$; assume for convenience $n_k = k$. For each k there exist a finite partition $\{E_{kj}\}_{j=1}^{n_k}$ of E_k and $\{x_{kj}\}_{j=1}^{n_k}$ with $\|x_{kj}\| \leq 1$ such that $a_k = \left\| \sum_{j=1}^{n_k} \mu_k(E_{kj}) x_{kj} \right\| \to \infty$. Set $v_k = (1/\sqrt{a_k})\mu_k)$. Now $v_k(f) \to 0$ for each $f \in \mathcal{E}(\Sigma, X)$ by $(2'')$. We may now apply Theorem 11 of [11] to obtain that the series $\sum_{k=1}^{\infty} \sum_{j=1}^{n_k} v_n(E_{kj}) x_{kj}$ converge uniformly with respect to n (the proof of Theorem 11 of [11] remains valid if the space $\mathcal{B}(\Sigma, x)$ is replaced by $\mathcal{E}(\Sigma, X)$). In particular,

$$\left\| \sum_{j=1}^{n_k} v_k(E_{kj}) x_{kj} \right\| = \sqrt{a_k} \to 0$$

contradicting $a_k \to \infty$.

REMARH 4. Theorem 3 can thus be regarded as a version of the Nikodym Boundedness Theorem for operator-valued measures. The statement of Theorem 1 for the case of vector-valued measures is equivalent to the statement that the space $\mathcal{S}(\Sigma)$ is a barrelled space. This suggests the following question: is the space $\mathcal{E}(\Sigma, X)$ barrelled? Theorem 3 seems to suggest that this may be the case; however, to traet this problem it seems necessary to have a description of $\mathcal{E}(\Sigma, X)$ or $\mathcal{B}(\Sigma, X)$). There does not seem to be such a description in the literature.

We now consider the Nikodym Convergence Theorem. We have the following version of theorem ([9], III. 7.4).

THEOREM 5. *Let $\mu: \Sigma \to X$ be countably additive. If*

$$(4) \qquad \lim \mu_n(E) = \mu(E) \textit{ exists for each } E \in \Sigma, \textit{ then}$$

$$(5) \qquad \mu \textit{ is countably additive and}$$

$$(6) \qquad \{\mu_n\}_{n=1}^{\infty} \textit{ is uniformly countably additive.}$$

Again the appropriate analogue of conditions (4) and (5) for operator measures in sca would be

(4′) $$\lim \mu_n(f) = \mu(f) \text{ exists for } f \in \mathcal{S}(\Sigma, X) \text{ and}$$

(5′) $$\mu \in \text{sca}.$$

The appropriate analogue for (6) would be

(6′) $$\{\mu_n\}_{n=1}^{\infty} \text{ is uniformly strongly bounded,}$$

where a sequence $\{\mu_n\}$ in sca is uniformly strongly bounded if $\sup_n \hat{\mu}_n(E_j) \to 0$ whenever $\{E_j\} \subseteq \Sigma$ decreases to the empty set. We give an example which shows that in general (4′) does not imply (5′) or (6′). For this we require the following criteria for uniform strong boundedness.

Proposition 6. *Let $\mu_n \in$ sca. Then (6') is equivalent to the condition* (6″) *for each disjoint sequence $\{E_j\} \subseteq \Sigma$ and bounded sequence $\{x_j\} \subseteq X$ the series $\sum_{j=1}^{\infty} \mu_k(E_j)x_j$ converge uniformly with request to n.*

Proof: If (6′) holds, the inequality $\| \sum_{j\in\sigma} \mu_n(E_j)x_j\| \le \hat{\mu}_n(\cup_{j\in\sigma} E_j \sup \|x_j\|$ shows (6″) holds.

If (6′) does not hold, we may assume there exist an $\varepsilon > 0$ and a disjoint sequence $\{E_j\}$ such that $\sup_n \hat{\mu}(E_j) > \varepsilon$ for all j. Thus, for each j there is an n_j such that $\hat{\mu}_{n_j}(E_j) > \varepsilon$. For each j there is a finite partition $\{E_{jk}\}_{k=Mj}^{Nj}$, of E_j and $\{x_{jk}\}_{k=Mj}^{Nj}, \|x_{jk}\| \le 1$, such that $\| \sum_{k=Mj}^{Nj} \mu_{nj}(E_{jk}x_{jk}\| > \varepsilon$, where we may assume $M_1 < N_1 < M_2 < N_2 < \cdots$. But then the series $\sum_{j=1}^{\infty} \sum_{k=Mj}^{Nj} \mu_n(E_{jk})x_{jk}$ doesn't converge uniformly with respect to n so that (6″) does not hold.

Remark 7. It follows from Proposition 6 and the proof of Theorem 3 that if $\{\mu_n\}$ is uniformly strongly bounded, then $\{\hat{\mu}_n(S)\}$ is bounded.

Example 8. Partition N into disjoint sets $\sigma_k = \{2^{k-1}, 2^{k-1}+1, \dots, 2^{k-1}\}$ and note each σ_k contains 2^{k-1} integers. If $j \in \sigma_k$, define $y_j \in c_0$ by $y_j = (1/2^{k-1})e_k$. For each k define $T_k \in L(l^1, c_0)$ by $T_k x = x_k y_k$, where $x = \{x_j\}$. For $n \in N$ and $\sigma \subseteq N$, set $\sigma \cap (n) = \sigma \cap \{1, \dots n\}$. Define measures $\mu_n: \mathcal{P} \to (l^1, c_0)$ by $\mu_n(\sigma) = \sum_{k\in\sigma(n)} T_k$. Each μ_n is countably additive in the uniform operator topology and $\hat{\mu}_n(\sigma) = 1$ for $\sigma \ne \emptyset$. Define $\mu: \mathcal{P} \to L(l^1, c_0)$ by $\mu(\sigma) = \sum_{k\in\sigma} T_k$. Since the series ΣT_k is subseries convergent with respect to the uniform operator topology, μ is countably additive with respect ot this topology and, moreover, μ has finite semi-variation with $\hat{\mu}(\sigma) = 1$ for $\sigma \ne \emptyset$. Now for $\sigma \subseteq N \mu_n(\sigma) \to \mu(\sigma)$ in norm but μ is not strongly bounded by the observation above, and, moreover, $\{\mu_n\}$ is not uniformly strongly bounded by Proposition 6 since the series $\sum_{n=1}^{\infty} \mu_j(n)e_n$ are not uniformly convergent. Again, we obtain an analogue of Theorem 5 by replacing the space $\mathcal{S}(\Sigma, X)$ in (4′) by $\mathcal{E}(\Sigma, X)$.

THEOREM 9. *Let* $\mu_n \in$ sca. *If*

$$\lim \mu_n(f) = \mu(f) \text{ exists for each } f \in \mathcal{E}(\Sigma, X), \tag{4''}$$

then $(5')$ *and* $(6')$ *hold.*

PROOF: The result follows from Theorem 11 of [11] (again noting that the proof of Theorem 11 of [11] is valid if the space $\mathcal{B}(\Sigma, X)$ is replaced by $\mathcal{E}(\Sigma, X)$).

REMARK 10. Theorem 9 can thus be regarded as a version of the Nikodym Convergence Theorem for operator measures in sca and again suggests that the space $\mathcal{E}(\Sigma, X)$ is the appropriate vector substitute for the space $\mathcal{S}(\Sigma)$.

A result closely related to the Nikodym Convergence Theorem is the Vitali-Hahn-Saks Theorem (VHS) ([9] III. 7.2). Using a result of Drewnowski ([7] 6.2) it is easily seen that we also have a version of VHS for measures in sca.

THEOREM 11. *Let* $\mu_n \in$ sca. *Suppose there is a positive, finite, countably additive measure* α *on* Σ *such that each* μ_n *is absolutely continuous with respect to* α. *Then* $(6')$ *holds iff*

$$\{\hat{\mu}_n\} \text{ is uniformly absolutely continuous with respect to } \alpha. \tag{7}$$

COROLLARY 12. *Let* $\mu_n \in$ sca *be such that* $\{\mu_n\}$ *is uniformly strongly bounded. Then there is a finite positive countably additive measure* α *on* Σ *such that* $\{\hat{\mu}_n\}$ *is uniformly absolutely continuous with respect to* α.

PROOF: For each μ_n there is a positive, finite, countably additive measure α_n such that $\hat{\mu}_n$ is absolutely continuous with respect to α_n ([4]* – Th). Set $\alpha = \sum_{n=1}^{\infty} \alpha/2^n(1 + \alpha_n(S))$ and apply Theorem 11.

As an application of the results above we consider the relationships between condition $(4')$ and conditions $(5')$, $(6')$.

PROPOSITION 13. *Let* $\mu_n \in$ sca *satisfy* $(4')$ *The following conditions are equivalent:*

(I) *conditions* $(5')$ *and* $(6')$

(II) $(4'')$

(III) $\lim \mu_n(f) = \mu(f)$ *exists for each* $f \in \mathcal{B}(\Sigma, X)$.

PROOF: Clearly (III) implies (II) and (II) implies (I) is just Theorem 8. It remains to show (I) implies (III). Let $f \in \mathcal{B}(\Sigma, X)$. Pick $\{\zeta_n\}$ simple such that $\zeta_n \to f$ pointwise on S with $\|\zeta_n(t)\| \le f(t)\| \le M$ for $t \in S$. Let α be the measure of Corollary 12. Let $\varepsilon > 0$. There is a $\delta > 0$ such that $\hat{\mu}_n(E) < \varepsilon$, $\mu(E) < \varepsilon$ whenever $\alpha(E) < \delta$, $E \in \Sigma$. Egoroff's Theorem implies there exest N and $E \in \Sigma$

such that $\|\zeta_N(t) - f(t)\| < \varepsilon$ when $t \in S\backslash E$ and $\alpha(E) < \delta$. From Remark 7 there is a $P > 0$ such that $\hat{\mu}_n(S) \leq P$ for all n and $\hat{\mu}(S) \leq P$. We thus have

$$
\begin{aligned}
\|\mu_n(f) - \mu(f)\| &\leq \left\| \int_S \zeta_N d\mu_n - \int_S \zeta_N d\mu \right\| + \left\| \int_S (f - \zeta_N) d\mu_n \right\| + \\
&+ \left\| \int_E (\zeta_N - f) d\mu \right\| + \left\| \int_{S\backslash E} (f - \zeta_N) d\mu_n \right\| + \left\| \int_{S\backslash E} (\zeta_N - f) d\mu \right\| \\
&\leq \left\| \int_S \zeta_N d\mu_n - \int_S \xi_N d\mu \right\| + 4M\varepsilon + 2P\varepsilon,
\end{aligned} \tag{8}
$$

and the first term on the right hand side of (8) goes to zero by (4′).

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