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ON MINIMAL SEPARATING BOOLEAN ALGEBRAS

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1. Introduction

Let x be a regular infinite cardinal and let X be a set. A family B consisting of subsets of X is called χ-algebra over X provided that it is closed under complementation and $< \chi$ union. A χ-algebra B is called a *separating* algebra if for any $x, y \in X$, $x \neq y$ there is $b \in B$ such that $x \in b$ but $y \notin b$. A separating χ-algebra B is called *minimal* if it is minimal in the sense of set inclusion. In [6] motivated by a problem in statistics, K. Namba constructed a minimal separating σ-algebra ($\equiv \aleph_1$-algebra) without atoms. Namely, let X be the set of all finite subsets of a set I and let $B_0(I)$ be the σ-algebra over X generated by sets of the form $\{F \in X \mid F \ni i\}$, $i \in I$. So, in [7] Namba proved that $B_0(I)$ is an atomless minimal separating σ-algebra over X if I is an uncountable set. In the same paper (see [6; p. 110] and also [7]) Namba asked for a general principle of creating atomless minimal separating σ-algebras which are *not* subalgebras of $B_0(I)$, for any I. We shall give here such a principle in proving the following theorem. $B_\chi(I)$ denotes the χ-algebra over $X = \{F \subseteq I \mid F \text{ is finite}\}$ generated by $\{F \in X \mid F \ni i\}$, where $i \in I$.

THEOREM A. *Let χ be a regular uncountable cardinal. Then there exists a family B_α, $\alpha < 2^\chi$ such that:*

(1) *B_α is an atomless minimal separating χ-algebra with χ-generators of power $\chi^{\overset{\chi}{\smile}}$,*

(2) *B_α is not a subalgebra of $B_\chi(I)$ for any I.*

(3) *If $f: B_\alpha \to B$ and $g: B_\beta \to B$ are χ-homomorphisms* 1*-to-*1 *and onto respectively, where B is a χ-algebra, then $\alpha = \beta$ and $f = g$.*

A χ-complete algebra B is said to be χ-hyper-rigid (see [3]) whenever for every χ-complete algebra B', every χ-complete homomorphisms f and g from B into B' such that f is 1-to-1 and g is onto we have $f = g$. The Theorem A answers a question of R. Bonnet (see [3; Problème 3]) who showed, assuming GCH, that for every infinite cardinal λ there exists a λ^+-hyper-rigid Boolean algebra of power λ^{+++}.

A tree T is a *normal* ω_1-tree (or simply ω_1-tree) iff: (1) height $(T) = \omega_1$; (2) $|R_0T| = 1$ and $|R_\alpha T| = \aleph_0$ for $0 < \alpha < \omega_1$; (3) if $x \in R_\alpha T$ and $\alpha < \beta < \omega_1$ then $|\{y \in R_\beta T \mid x <_T y\}| = \aleph_0$ (see [5]). In [5; p. 70], T. Jech asked for a rigid normal ω_1-tree. U. Avraham [1], J.E. Baumgartner (unpublished) and the author [9], all independently, constructed such a tree which is also Aronszajn. In [2], U. Avraham and S. Shelah constructed a model of set theory in which for every two (normal) Aronszajn trees T and T' there exists a closed and unbounded (club) set $C \subseteq \omega_1$ such that $T \mid C$ and $T' \mid C$ are isomorphic. So, in that model any Aronszajn tree is not "really" rigid. Hence, it is natural to ask for the existence (in ZFC) of two "really" non-isomorphic ω_1-trees and for a "really" rigid ω_1-tree. We shall extract from the proof of Theorem A the following strong answer to this question.

THEOREM B. *There is a family T_α, $\alpha < 2^{\aleph_1}$ such that:*

(1) *T_α is a normal ω_1-tree;*

(2) *If C is a club in ω_1 and if $f: T_\alpha|C \to T_\beta|C$ is* 1*-to-*1 *order and level preserving then $\alpha = \beta$ and $f = id$.*

We use the usual notation and conventions. The basic definitions can be found in any standard text in set theory.

2. Representation tbeorem for minimal χ-algebras

From now $\chi \geq \aleph_0$ is a fixed regular cardinal. A topological space X is χ-*additive* if the intersection of any $< \chi$ open sets of X is open in X. The χ-*topology* on 2^I (or on a subset of 2^I) is the smallest χ-*additive* topology containing the Tychonov topology on 2^I (or on a subset of 2^I). For $x \in 2^I$ define $\operatorname{supp}(x) = \{i \in I \mid x(i) = 1\}$ and $X_\chi(I) = \{x \in 2_I / \operatorname{supp}(x) \text{ is finite}\}$ with the χ-topology. Then $X_\chi(I)$ is a χ-compact χ-additive space (see [6]). (A space X is χ-compact iff any open cover of X contains a subcover of cardinality $< \chi$). It is easy to see that the Boolean algebra of all closen subsets of $X_\chi(I)$ is a minimal separating χ-algebra over $X_\chi(I)$ isomorphic to the algebra $B_\chi(I)$ defined in § 1.

Since we intend to prove the Theorem A in its dual form we need the following theorem parts of which are proved in [6] and [8].

THEOREM C. *The following statements are equivalent:*

(i) *B is a minimal separating χ-algebra over X;*

(ii) *every proper χ-additive ideal of B is contained in a χ-additive prime ideal of the form $\{b \in B \mid b \not\ni x\}$, $x \in X$;*

(iii) *X with the topology generated by B is a χ-compact χ-additive space and B is equal to the field of all closen subsets of X.*

Let $St(B)$ be the Stone space of a minimal separating χ-algebra B over X. Let $St_x(B) = \{I \in St(B) \mid I$ is a χ-additive prime ideal of $B\}$. Then by the Theorem C we conclude that X, with the topology generated by B, is homeomorphic to $St_x(B)$ with the topology induced from $St(B)$, by the homeomorphism $x \to \{b \in B \mid b \not\ni x\}$. Hence, to answer the above mentioned Namba's question we have to find a general principle to create χ-compact χ-additive spaces (without isolated points) which are not included in the continuouos image of any space of the form $X_\chi(I)$.

3. The construction

Let χ be a regular uncountable cardinal fixed from now on. Let us make the following conventions. 2^χ is the set $\{f \mid f\colon \chi \to 2\}$ or is the cardinality of this set. All spaces are regular Hausdorff. All mappings between spaces are continuous and between algebras are homomorphisms. We shall use the following well known fact without mention: If X is a χ-compact χ-additive space, Y a χ-additive space, and if $f\colon X \to Y$ is a 1-to-1 mapping, then X is homeomorphic to the closed subspace $f''X$ of Y.

Let S be a stationary subset of χ. A topological space X is *S-compact* if for every stationary $S' \subseteq S$ and every sequence $\langle x_\delta \mid \delta \in S' \rangle$ from X there are $x \in X$ and stationary $S'' \subseteq S$ such that $\langle x_\delta \mid \delta \in S'' \rangle$ converges to x, i.e. for every open $U \ni x$ there exists $\delta' \in S''$ such that $\delta' \leq \delta \in S''$ implies $x_\delta \in U$. Since S-compactness is not defined for S nonstationary in χ, let us agree that whenever we mention S-compactness, S is a stationary subset of x. The proofs of the following two lemmas are easy.

LEMMA 3.1. *If Y is included as a closed subset of a continuous image of X then the S-compactness of X implies the S-compactness of Y.*

LEMMA 3.2. *The space $X_\chi(I)$ (see §2) is S-compact for every stationary $S \subseteq \chi$.*

Let $\Omega = \{\delta < \chi \mid cf(\delta) = \omega\}$. Fix a strictly increasing sequence $\eta_\sigma = \langle \eta_\sigma(n) \mid n < \omega \rangle$ of ordinals cofinal with δ, for every $\delta \in \Omega$. Now, for every $S \subseteq \Omega$ we define

$$X(S) = \{x \in 2^\chi \mid \mathrm{supp}\,(x) \text{ is finite or for some } \sigma \in S,\ \mathrm{supp}\,(x) - \sigma \text{ is finite and } \mathrm{supp}\,(x) \cap \sigma = \{\eta_\sigma(n) \mid n < \omega\}\}.$$

We shall consider $X(S)$ as a subspace of 2^χ with the χ-topology. From now on, any set of ordinals we are working with, will be included in Ω without mention. Note that if $S' \subseteq S$ then $X(S')$ is a closed subspace of $X(S)$ and that $X(\emptyset) = X_\chi(\chi)$. Also note that $X(S)$ has no isolated points.

LEMMA 3.3.

(a) *$X(S)$ is a χ-compact χ-additive space for every S.*

(b) *$X(S)$ is S'-compact iff $S \cap S'$ is nonstationary in χ.*

(c) *If $f\colon X(S) \to X(S')$ is* 1*-to-*1 *then $S - S'$ is nonstationary in χ.*

(d) *If $f\colon X(S) \to X(S')$ is onto then $S' - S$ is nonstationary in χ.*

Proof: (a) It is well known (see [8]) that for proving χ-compactness of $X(S)$ it is enough to prove that every sequence $\langle x_\sigma \mid \sigma < \chi\rangle$ from $X(S)$ has a convergent subsequence. We may assume $x_{\delta'} \neq x_\delta$, for $\delta \neq \delta'$.

Case I. $S_1 = \{\delta \in \Omega \mid \mathrm{supp}\,(x_\sigma) \cap \delta$ is bounded in $\delta\}$ is stationary.

This case is similar and less complicated than the following case. (Use the Pressing Down Lemma (PDL); see [4].)

Case II. S_1 is nonstationary in χ.

We may assume $S_1 = \emptyset$, i.e. that $\delta \in \Omega$ implies $\mathrm{supp}\,(x_\delta) \cap \delta = \{\eta_\delta(n) \mid n < \omega\}$. Let $n = \min\{m \mid \{\eta_\delta(m) \mid \delta \in \Omega\}$ is unbounded in $\chi\}$. So, we can find an unbounded set $N \subseteq \chi$ and finite $F \subseteq \chi$ such that $\delta < \delta'$, δ, $\delta' \in N$ implies $F < \eta_\delta(n) < \eta_{\delta'}(n)$ and $F = \{\eta_\delta(m) \mid m < n\} = \{\eta_{\delta'}(m) \mid m < n\}$. Let $x \in X(S)$ be defined by $\mathrm{supp}\,(x) = F$. Then $\langle x_\delta \mid \delta \in N\rangle$ converges to x.

(b) Assume first that $S \cap S'$ is nonstationary in χ and prove that $X(S)$ is S'-compact. Let $S_0 \subseteq S'$ be a stationary set and let $\langle x_\delta \mid \delta \in S_0\rangle$ be a sequence from $X(S)$. We may assume $S_0 \cap S = \emptyset$. Hence $\mathrm{supp}\,(x_\delta) \cap \delta$ is bounded in δ for every $\delta \in S_0$. So for every $\delta \in S_0$ there is an $\varepsilon(\delta) < \delta$ such that $\mathrm{supp}\,(x_\delta) \cap \varepsilon(\delta) = \{\eta_{\varepsilon(\delta)}(n) \mid n < \omega\}$ and $\mathrm{supp}\,(x_\delta) - \varepsilon(\delta)$ is finite. By the PDL there are stationary $S_1 \subseteq S$, $\varepsilon_0 \in S$ and finite $F \subseteq [\varepsilon_0, \chi)$ such that $\varepsilon(\delta) = \varepsilon_0$ and $\mathrm{supp}\,(x_\delta) \cap [\varepsilon_0, \delta) = F$, for every $\delta \in S_1$. Define $x \in X(S)$ by

$$\mathrm{supp}\,(\chi) = F \cup \{\eta_{\varepsilon_0}(n) \mid n < \omega\}.$$

Clearly $\langle x_\delta \mid \delta \in S_1\rangle$ converges to x.

Assume now that $S_0 = S \cap S'$ is stationary in χ. For $\delta \in S_0$, we define $x_\delta \in X(S)$ by $\mathrm{supp}\,(x_\delta) = \{\eta_\delta(n) \mid n < \omega\}$. Then an easy application of the PDL shows that for no stationary $S_1 \subseteq S_0$, $\langle x_\delta \mid \delta \in S_1\rangle$ is a convergent sequence in $X(S)$.

(c) Follows directly from (a), (b) and the Lemma 3.1.

(d) Follows directly from (b) and the Lemma 3.1. This completes the proof of Lemma 3.3.

Let $S \subseteq \Omega$ be a stationary set in χ. Then, by the Lemmas 3.2. and 3.3, $X(S)$ is a χ-compact χ-additive space which is not included in the continuous image of any space of the form $X_\chi(I)$. Thus, if $A(S)$ is the Boolean algebra of all clopen subsets of $X(S)$, then $A(S)$ is an atomless minimal separating χ-algebra over $X(S)$ which is not a subalgebra of any algebra of the form $B_\chi(I)$. Moreover, the family $\{A(S) \mid S \subseteq \Omega$ is stationary$\}$ has other interesting properties (see Lemma 3.3 (c) and (d)). This gives a quite complete answer to Namba's question.

It is well known that there is a family $K \subseteq \{S \subseteq \Omega \mid S$ it stationary$\}$ such that $|K| = 2^\chi$ and S, $S' \in L$, $S \neq S'$ implies $S - S'$ is stationary in χ (see, e.g. [11]). So, the family $A(S)$, $S \in K$ satisfies the Theorem A except the last condition in (3). To get this condition we need a little more work.

4. Proof of Theorem A

Let $S \subseteq \Omega$ be a fixed stationary set. Since we can easily decompose S into χ many stationary sets we can also construct a sequence $\langle S_\delta \mid \delta \in \{0\} \cup S\rangle$ of stationary subsets of S such that

(i) $\delta < \min S_\delta$;

(ii) $S_\delta \cap S_{\delta'}, = \emptyset$ for $\delta \neq \delta'$;

(iii) $\cup\{S_\delta \mid \delta \in \{0\} \cup S\} = S$.

Now for every $\delta \in S$, fix a strictly increasing sequence $\eta_\delta = \langle \eta_\delta(n) \mid n < \omega\rangle$ of ordinals converging to δ such that:

(iv) if $\delta \in S_{\delta'}$, then $\delta' < \eta_\delta(0)$ for every $\delta, \delta' \in \{0\} \cup S$;

(v) for every $\delta \in \{0\} \cup S$ and every finite $F \subseteq \chi - \delta$ there exists $n < \omega$ such that $\{\gamma \in S_\delta | \{\eta_\gamma(m) | m < n\} = F\}$ is a stationary subset of χ.

Finally, let

$$\begin{aligned} Y(S) = \{x \in 2^\chi \mid &(\text{there exists } \{\delta_0, \dots, \delta_n\} \subseteq \{0\} \cup S \text{ such that } \delta_0 = 0,\, \delta_{i+1} \in S_{\delta_i}; \\ &\text{for } i < n < \omega,\ \operatorname{supp}(x) \cap \delta_n = \{\eta_{\delta_i}(m) \mid m < \omega,\ 1 \le i \le n\} \text{ and} \\ &\operatorname{supp}(x) - \delta_n \text{ is finite}\}. \end{aligned}$$

We consider $Y(S)$ as a subspace of 2^χ with the χ-topology.

The proof of the following lemma is almost identical to the proof of Lemma 3.3.

Lemma 4.1.

(a) *$Y(S)$ is χ-compact for every S.*

(b) *$Y(S)$ is S'-compact iff $S \cap S'$ is nonstationary in χ*

(c) *If $f: Y(S) \to Y(S')$ is 1-to-1 then $S - S'$ is nonstationary in χ.*

(d) *If $f: Y(S) \to Y(S')$ is onto then $S' - S$ is nonstationary in χ.*

Let $K \subseteq \{S \subseteq \Omega \mid S \text{ is stationary}\}$ be a fixed family such that $|K| = 2^\chi$ and $S, S' \in K$, $S \neq S'$ implies $S - S'$ is a stationary subset of χ. The following is the dual form of Theorem A.

Theorem D. *The family $Y(S)$, $S \in K$ has the following properties, where $S, S' \in K$:*

(1) *$Y(S)$ is a χ-compact χ-additive space.*

(2) *$Y(S)$ is not included in the continuous image of any space of the form $X_\chi(I)$.*

(3) *If $f: Y \to Y(S)$ and $g: Y \to Y(S')$ are 1-to-1 and onto, respectively, where Y is a χ-additive space, then $S = S'$ and $f = g$.*

Proof: (1) and (2) follow from the Lemmas 3.2 and 4.1. Let us prove (3). Since Y is χ-compact, Y is homeomorphic to the closed subspace $f''Y$ of $Y(S)$. Hence Y is $(\Omega - S)$-compact (Lemmas 3.1 and 4.2). Since g is onto, $Y(S')$ is also $(\Omega - S)$-compact (Lemma 3.1). By the property of the family K we must have $S = S'$ by the Lemma 4.2 (b).

Let us prove that $f''Y = Y(S)$, i.e. that Y is homeomorphic to $Y(S)$. Otherwise we can find $\xi < \chi$ and $t{:}\,\xi \to 2$ such that the basic open set $Y(S)^t = \{x \in Y(S) \mid t \subset x\}$ is non-empty and disjoint from $f''Y$. By the construction of $Y(S)$ it is easily seen that $Y(S)^t$ has the structure similar to $Y(S^t)$ for some stationary set $S^t \subseteq S$ (which is uniquely determined by t). Moreover, $Y(S) - Y(S)^t$ is a S^t-compact space. Hence Y is a S^t-compact space. Since $g{:}\,Y \to Y(S') = Y(S)$ is onto, $Y(S)$ is also a S^t-compact space, a contradiction (Lemma 4.1 (b)). Hence f is a homeomorphism from Y onto $Y(S)$. Let $h = gf^{-1}{:}\,Y(S) \to Y(S)$. Then h is onto. Assume $h \neq id$. Then we can easily find $t{:}\,\xi \to 2$ and $u{:}\,\xi \to 2$ $(\xi, \zeta \in x)$ such that $Y(S)^t$ and $Y(S)^u$ are disjoint non-empty and $Y(S)^u$ is included in $h''Y(S)^t$. By the construction of $Y(S)$ there are uniquely determined disjoint stationary sets S^t, $S^u \subseteq S$ such that $Y(S)^t$ and $Y(S)^u$ have the structures similar to $Y(S^t)$ and $Y(S^u)$, respectively. So again we have a contradiction (Lemmas 3.1 and 4.1 (b)).

This completes the proof of Theorem D.

Let $S \in K$ and let $B(S)$ be the Boolean algebra of all clopen subsets of $Y(S)$. Then $B(S)$, $S \in K$ satisfies the Theorem A using the Theorem D and the "χ-Stone duality".

REMARK 4.2. *Let $S \subseteq \Omega$ be a stationary set in χ and let $\prec$ be the lexigraphical ordering of $Y(S)$. Let $C(S)$ be the Boolean algebra of all finite unions of $\prec$-intervals of the form $[x, y), x, y \in Y(S) \cup \{-\infty, +\infty\}$. Then $C(S)$ is a very strongly rigid Boolean algebra (see* [10], [11]*; very strongly rigid $\equiv \aleph_0$-hyper rigid, see* §1*).*

5. Proof of Theorem B.

In this section we consider the case $\chi = \aleph_1$ of the previous constructions.

Let $S \subseteq \Omega = \{\alpha < \omega \mid \alpha \text{ is a limit ordinal} > 0\}$ be stationary set. Then $Y(S)$ naturally determines an ω_1-tree $T(S) = \{t \mid t \subsetneqq x \text{ for some } x \in Y(S)\}$, i.e.,

$T(S) = \{t : \alpha \to 2 \mid \alpha < \omega_1$ and there exist $n < \omega$ and $\{\delta_0, \dots, \delta_n\} \subseteq \{0\} \cup S$ such that $\delta_0 = 0$, $\delta_{i+1} \in S_{\delta_i}$ for $i < n$, $\operatorname{supp}(t) \cap \delta_n = \{\eta_{\delta_i}(m) \mid m < \omega$ and $1 \leq i \leq n\}$ and $\operatorname{supp}(t) - \delta_n$ is finite$\}$.

Let $S, S' \subseteq \Omega$ be stationary sets in ω_1, let C be a closed and unbounded subset of ω_1, and let $h : T(S) \mid C \to T(S') \mid C$ be a 1-to-1 order and level preserving map. Define $f : Y(S) \to Y(S')$ by $f(x) =$ the element of $Y(S')$ uniqely determined by the ω_1-chain $\{h(t) \mid t \in T(S) \mid C \text{ and } t \subset x\}$ of $T(S')$. Clearly f is a well-defined 1-to-1 mapping. It is easily seen that f is also a continuous mapping. Now the Theorem B follows from the case $\chi = \aleph_1$ of the dual form of Theorem D.

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