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STATIONARY SETS TREES AND CONTINUUMS

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The Suslin problem proposed in 1920 by M. Suslin has been very stimulating. The first advance on the problem was made by Đ. Kurepa in [K1] where he proved the equivalence of the existence of Suslin continuum ($\equiv$ a non-separable linearly ordered continuum with no uncountable family of disjoint open intervals) and Suslin tree ($\equiv$ an uncountable tree with no uncountable chains nor antichains). Since the construction of either the continuum or the tree seemed to be very hard, the above equivalence suggested the construction of an uncountable tree with countable levels and with no uncountable chains. Surprisingly, such a construction was indeed possible; this was done by N. Aronszajn in [K1; p 96]. But the above equivalence (in fact, its proof) also suggested the construction of a linearly ordered first countable continuum which has no dense set equal to the union of countably many discrete subspaces (see [K2]). We shall give such a construction in this paper.

It turns out that such a construction also solves several other problems mostly from general topology (see Remark 4.7). Namely, in various situations the Suslin continuum was used as a non-absolute (i.e., not in ZFC) counterexample, and it was then asked whether an absolute counterexample exists. Our continuum is such a counterexample.

The paper is organized as follows. In § 1 we list some definitions and notations. In § 2 we define stationarity with respect to ω_1-trees. This gives us a possibility of assigning to every ω_1-tree T the ideal I_T of all subsets of ω_1 which are nonstationary w.r. to T. In the same section we consider some properties of the function $T \to I_T$ and prove that I_T is a normal ideal for every T; that T is special iff $I_T = \mathcal{P}(\omega_1)$, etc. In § 3 we compute I_T for some kind of trees. In § 4 we assign to every subset S of ω_1 a linearly ordered continuum $C(S)$ and consider some properties of the function $S \to C(S)$. If S is a bistationary subset of ω_1, then $C(S)$ is a first countable continuum which has no dense set equal to the union of countably many discrete subspaces ($\equiv$ has no σ-disjoint π-base). In § 5 we prove, e.g., that if S and S' are disjoint stationary subsets of ω_1, then $C(S)$ is not homeomorphic to a subspace of $C(S')$. This gives us a possibility of constructing many (strongly) rigid first countable linearly ordered continuums.

A part of these results was announced in [T1].

1. Definitions and notations

We work in ZFC set theory and adapt the usual notation and conventions. The basic definitions can be found in any standard text in set theory and topology.

A *tree* is a poset $(T, <_T)$ such that $\hat{t} = \{s \in T \mid s <_T t\}$ is well ordered by $<_T$ for every $t \in T$. $\gamma t = tp(\hat{t}, <_T)$ is the *height* of t in T. We shall often identify $(T, <_T)$ with its domain T. The $\alpha'th$ *level* of T is the set $R_\alpha T = \{t \in T \mid \gamma t = \alpha\}$. The height of T is the ordinal $\gamma T = \min\{\alpha \mid R_\alpha T = \emptyset\}$. If $\gamma T = \alpha$ then T is an α-*tree*. An α-tree T is *normal* if for every $\beta < \beta' < \alpha$ and $t \in R_\beta T$ there exist $s, s' \in R_{\beta'} T$ such that $s \neq s'$ and $t <_T s, s'$. In this paper we shall consider only trees of height $\leq \omega_1$. If X is a set of ordinals then by $T \mid X$ we denote subset $\cup\{R_\alpha T \mid \alpha \in X\}$ of T under the inherited ordering. If $t \in T$ and $\alpha \leq \gamma t$ then by $t \mid \alpha$ we denote a unique $s \in R_\alpha T$ such that $s \leq_T t$. We shall always assume that our trees have the least element $\emptyset$. Let U be a subset of T then $f : U \to T$ is *regressive* if $f(t) <_T t$ for every $t \in U - \{\emptyset\}$. T is a special tree if there exists $h : T \to \omega$ such that $s <_T t$ implies $h(s) \neq h(t)$. Let $(T_i, <_i)$, $i \in F$ be a family of trees then $\otimes\{T_i \mid i \in F\}$ denotes the set $\{\langle t_i \mid i \in F\rangle \mid$ there is an α such that $t_i \in R_\alpha T_i$ for every $i \in F\}$ ordered by: $\langle s_i \mid i \in F\rangle <' \langle t_i \mid i \in F\rangle$ iff $s_i <_i t_i$ for every $i \in F$. Clearly $(\otimes\{T_i \mid i \in F\}, <')$ is a tree. If C is a closed and unbounded subset of ω_1 and if $f : T \mid C \to T' \mid C$ is an order preserving, level preserving one-to-one mapping then T is *embeddable on a club* in T'. U is dense in T if for every $t \in T$ there exists an $s \in U$ such that $t \leq_T s$.

The following definitions are common: *ideal* (on ω_1), *nonprincipal ideal*, *σ-complete ideal* … . If $\langle X_\alpha \mid \alpha < \omega_1\rangle$ is a sequence of subsets of ω_1 then the *diagonal union* of the sequence, denoted $\nabla\langle X_\alpha \mid \alpha < \omega_1\rangle$, is defined to be $\{\beta < \omega_1 \mid$ for some $\alpha < \beta$, $\beta \in X_\alpha\}$. I is *normal* ideal on ω_1 if $\nabla\langle X_\alpha \mid \alpha < \omega\rangle \in I$ for every sequence $\langle X_\alpha \mid \alpha < \omega_1\rangle$ of elements of I. We say that a set $S \subseteq \omega_1$ is *stationary* in ω_1 if $S \cap C \neq \emptyset$ for every club $C \subseteq \omega_1$. S is *costationary* in ω_1 if $\omega_1 - S$ is stationary in ω_1. S is *bistationary* if both S and $\omega_1 - S$ are stationary; NS denotes the ideal of all nonstationary subsets of ω_1. If $S \subseteq \omega_1$ then I_s denotes the ideal $\{X \subseteq \omega_1 \mid X \cap S \in NS\}$.

A topological space Y is *first countable* if every point of Y has a countable neighbourhood base in Y. A family $\mathcal{P}$ of nonempty open sets of Y is a π-*base* of Y if every non empty open set contains a member of $\mathcal{P}$. A family $\mathcal{P}$ is called σ-*disjoint* if $\mathcal{P} = \cup\{\mathcal{P}_n \mid n < \omega\}$ where each $\mathcal{P}_n$ is a disjoint family. Y is a K_0-*space* iff there exists a countable family of discrete subspaces of Y, the union of which is dense in Y. Y is a *Blumberg* space if for any real-valued function f on Y, there exists a dense subspace $D \subseteq Y$ such that $f \mid D$ is continuous. All spaces considered are Hausdorff.

2. Nonstationarity with respect to trees

Definition 2.1. Let T be an $(\leq \omega_1-)$ tree and let $X \subseteq \omega_1$. X is *nonstationary in* T if there exists a regressive mapping $f : T \mid X \to T$ such that $f^{-1}(t)$ is a special subtree of T for every $t \in T$.

If $T = \omega_1$ then according to a well known Neumer theorem [N], which is a generalisation of a result of Alexandroff and Urysohn [AU], we have the usual definition of nonstationarity in ω_1.

For every ω_1-tree T define $I_T = \{X \subseteq \omega_1 \mid X \text{ is nonstationary in } T\}$. Then I_T is a σ-complete ideal on ω_1. If $X \subseteq \omega_1$ is nonstationary in ω_1 then X is nonstationary in T (i.e. $X \in I_T$) for every ω_1-tree T. Namely, let $C \subseteq \omega_1$ be closed and unbounded such that $C \cap X = \emptyset$. Define $f : T \mid X \to T$ by $f(t) = t \mid \alpha(t)$ where $\alpha(t) = \max(C \cap \gamma t)$. Then f is a regressive mapping which examplifies $X \in I_T$ as it is easy to see.

Let us now mention some properties of the ideals of the form I_T.

Theorem 2.2.

(i) *I_T is a normal ideal on ω_1 for every T;*

(ii) *If T is embeddable on a club in T' then $I_{T'} \subseteq I_T$. Especially if T is an initial part of T' then $I_{T'} \subseteq I_T$;*

(iii) $I_{\otimes\{T_i \mid i \in F\}} \supseteq \cup\{I_{T_i} \mid i \in F\}$.

Proof: (i) Let $\langle X_\alpha \mid \alpha < \omega_1\rangle$ be a sequence from I_T and let $X = \nabla\langle X_\alpha \mid \alpha < \omega_1\rangle$ be the diagonal union of the sequence. We have to prove $X \in I_T$. For $\beta \in X$ define $\alpha(\beta) = \min\{\alpha \mid \beta \in X_\alpha\}$. Then $\alpha(\beta) < \beta$ for every $\beta \in X$. Let $f_\alpha : T \mid X_\alpha \to T$ examplifies $X_\alpha \in I_T$ for every $\alpha < \omega_1$. Let $j : \omega_1 \times \omega_1 \to \omega_1$ be a bijection such that $j(\alpha, \alpha') = \beta$ implies $\alpha' \leq \beta$ and let $j_0, j_1 : \omega_1 \to \omega_1$ are defined by $j(j_0(\alpha), j_1(\alpha)) = \alpha$. Let $C = \{\alpha < \omega_1 \mid \text{if } \beta, \beta' < \alpha \text{ then } j(\beta, \beta') < \alpha\}$. It is easy to see that C is closed and unbounded in ω_1. Define $f : T \mid (X \cap C) \to T$ by $f(t) = t \mid j(\alpha(\gamma t), \gamma f_{\alpha(\gamma t)}(t))$. Then f is regressive. Let us prove that f examplifies $X \cap C \in I_T$. Let $u \in T$, then $f^{-1}(u) \subseteq f^{-1}_{j_0(\gamma u)}(u \mid j_1(\gamma u))$ as is easily seen from the definition of f (remember $j_1(\gamma u) \leq \gamma u$ so that $u \mid j_1(\gamma u)$ is well defined). Since $f^{-1}_{j_0(\gamma u)}(u \mid j_1(\gamma u))$ is special then so is $f^{-1}(u)$. So $X \cap C \in I_T$, hence $X = (X \cap C) \cup (X - C) \in I_T$ since $NS \subseteq I_T$.

(ii) is travial. Let us prove (iii). Let $X \in I_{T_{i_0}}$ for some $i_0 \in F$ and let $f_{i0} : T_{i0} \mid X \to T_{i_0}$ examplifies this fact. Define $f : \otimes\{T_i \mid i \in F\} \mid X \to \otimes(T_i \mid i \in F)$ by $f\langle t_i \mid i \in F\rangle = \langle t_i|\alpha|i \in F\rangle$ where $\alpha = \gamma f_{i0}(t_{i0})$. Then f is regressive. It is easy to see that f examplifies $X \in I_{\otimes\{T_i \mid i \in F\}}$.

Remark 2.3. (1) Theorem 2.2 (i) is a generalization of well known Fodor theorem [F] which asserts that the ideal of nonstationary in ω_1 subsets of ω_1 is normal.

(2) If T has an ω_1-branch then $I_T = NS$ by (ii) of the theorem 2.2. The converse does not hold (see theorem 3.2 (ii)).

THEOREM 2.4. *T is special iff $I_T = \mathcal{P}(\omega_1)$.*

Proof. The direct implication is trivial. Assume $IT = \mathcal{P}(\omega_1)$, i.e. $\omega_1 \in I_T$ and let $f : T \to T$ examplifies this fact, i.e. f is a regressive mapping such that $f^{-1}(t)$ is special subset of T for every $t \in T$. Let $g_t : f^{-1}(t) \to \omega$ be a specializing maping for every $t \in T$. Define $h : T - \{\emptyset\} \to [\omega]^{<\omega}$ by $h(t) = (n_0, \dots, n_{k-1})$ where $(n_0, \dots, n_{k-1})$ is determined as follows. Since f is regressive there exists a unique sequence $(t_0, \dots, t_k)$, $\emptyset = t_k <_T t_{k-1} <_T t_{k-2} \dots <_T t_0 = t$ such that $f(t_i) = t_{i+1}$ for every $i < k$. Then $n_i = g_{t_{i+1}}(t_i)$ for every $i < k$. Now is quite easy to check that h specializes T.

3. Trees

DEFINITION 3.1. Let E be a linearly ordered set. By σE (see [K4]) we denote the set of all bounded well ordered subsets of E ordered as follows: $s < t$ iff s is a proper initial segment of t. σE is clearly a tree. Since in this paper we are not interested in trees of height $> \omega_1$ we shall always assume that E contains no uncountable well ordered subsets, i.e., that the height of σE *is* $\leq \omega_1$.

3.2. Let S be a subset of ω_1. Then by $T(S)$ we denote the set of all countable closed in ω_1 subsets of S ordered by the relation $<=$ "*is* a proper initial part of" (as above). Then $T(S)$ is a tree of height $\leq \omega_1$.

In this section we shall consider the trees of the form σE and $T(S)$. Especially we shall find $I_{\sigma E}$ and $I_{T(S)}$.

THEOREM 3.3.

(i) $I_{T(S)} = I_S$ *for every* S*;*

(i) $I_{\sigma E} = \mathcal{P}(\omega_1)$ *or* NS *according to whether* E *is scattered or not.*

Proof: (i) Let $X \in I_S$. Since $I_{T(S)}$ contains NS we can assume $X \cap S = \emptyset$ and $\lim(\alpha)$ for every $\alpha \in X$. Define $f : T(S) \mid X \to T(S)$ by $f(t) = t \cap \gamma t$. Clearly f is regressive (since $\lim(\gamma t)$ and $\gamma t \notin S$). Let us prove that $f^{-1}(t)$ is special for every t, so that we can conclude $X \in I_{T(S)}$. Let $s \in R_0 f^{-1}(t) =$ the set of all minimal elements of $f^{-1}(t)$ and let $u \in f^{-1}(t)$, $s < u$. Then γs, $\gamma s \in X$, $\gamma s < \gamma u$ and $t = s \cap \gamma s = u \cap \gamma u$. Hence $\gamma u \leq \min(s - t)$. So subtree $\{u \in f^{-1}(t) \mid s \leq u\}$ has height $\leq \min(s - t) + 1$. This proves that $f^{-1}(t)$ is the incomparable union of trees of height $< \omega_1$ and so $f^{-1}(t)$ is special.

Assume now $X \in I_{T(S)}$ and prove $X \in I_S$. Assume otherwise, i.e. that $X \cap S$ is stationary in ω_1. Let $f : T(S) \mid X \to T(S)$ be a regressive mapping such that $f^{-1}(t)$ is special for every $t \in T(S)$. Let $g_t : f^{-1}(t) \to \omega$ specialize $f^{-1}(t)$ for every $t \in T$.

Define, by induction, a sequence $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ of countable subscts of $T(S)$ and a normal sequence $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$ of ordinals in ω_1 as follows:

$A_0 = \{\emptyset\}$, $\delta_0 = 0$. lf $\lim(\alpha)$ then $A_\alpha = \cup\{A_\beta \mid \beta < \alpha\}$ and $\delta_\alpha = \sup\{\delta_\beta \mid \beta < \alpha\}$.

Now suppose that A_α and δ_α are defined. For each $t \in A_\alpha$ $s \leq t$ and $i < \omega$ let $u(t, s, i) \in T(S)$ be such that $t \leq u(t, s, i) \in f^{-1}(s)$, $g_s(u(t, s, i)) = i$ and

$\max(u(t,s,i)) > \delta_\alpha$ if such $u(t,s,i)$ exists, otherwise let $u(t,s,i) = t$. Let $A_{\alpha+i} = A_\alpha \cup \{u(t,s,i) \mid s \leq t \in A_\alpha, i < \omega\}$, $A_{\alpha+1} = \{s \in T(S) \mid s \leq t$ for some $t \in A_{\alpha+1}\}$ and $\delta_{\alpha+1} = \sup\{\max(t) \mid t \in A_{\alpha+1}\}$.

Since $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$ is normal and $X \cap S$ is stationary there exists $\alpha < \omega_1$ such that $\delta_\alpha = \alpha \in X \cap S$.

Let $\langle \alpha_n \mid n < \omega \rangle$ be increasing and cofinal in α and let $p : \omega \times \omega \to \omega$ be such a bijection that $p(i,j) = k$ imply $i \leq k$. Define, by induction, sequences $\langle s_n \mid n < \omega \rangle$ and $\langle t_n \mid n < \omega \rangle$ of elements of $T(S)$ such that $\langle t_n \mid n < \omega \rangle$ is increasing and $s_n \leq t_n$ for every n as follows. Let $s_0 = t_0 = \emptyset$. Now suppose that s_n, t_n are defined. Let $p(i,j) = n$. Let $t_{n+1} \in A_\alpha$ be such that $t_n < t_{n+1}$, $\max(t_{n+1}) > \alpha_n$, $f(t_{n+1}) = s_i$ and $g_{s_i}(t_{n+1}) = j$ if such t_n exists, otherwise chose $t_{n+1} \in A_\alpha$ such that $t_n \leq t_{n+1}$ and $\max(t_{n+1}) > \alpha_n$ (it is easy to see that such t_{n+1} exists). Choose $s_{n+1} \leq t_{n+1}$ in such a way that at the end $\{s_m \mid m < \omega\} = \{s \in T(S) \mid s \leq t_n$ for some $n\}$ holds. Let $t = (\cup\{t_n \mid n < \omega\}) \cup \{\alpha\}$. Then $t \in T(S) \mid X$. Hence $s = f(t) < t$ and there exists an $i < \omega$ such that $s = s_i$. Let $j = g_s(t)$ and let $n = p(i,j)$. Since $s \in A_\alpha$ there exist $m \geq n$ such that $s \in A_{\alpha_m}$. By the definition of $A_{\alpha_{m+1}}$ and the property of t we know that $u = u(t_n, s, j) \in A_{\alpha_{m+1}} \subseteq A_\alpha$ has properties $t_n \leq u$, $\max(u) > \alpha_n$, $f(u) = s_i = s$ and $g_s(u) = j$. So, t_{n+1} also has these properties. Especialy $f(t_{n+1}) = s$ and $g_s(t_{n+1}) = j$ which is a contradiction since $t_{n+1} < t$, $f(t) = s$ and $g_s(t) = j$. This completes the proof of (i).

(ii) If E is scattered then σE is special by [T 2; Corollary 2), hence $I_{\sigma E} = \mathcal{P}(\omega_1)$. If E is not scattered then E contains a copy of Q (the rationals) and then as it is easy to see σE contains an initial part isomorphic to σQ. So by theorem 2.1 (ii) it is enough to prove $I_{\sigma Q} = NS$. The proof of this fact is similar to the proof of the direct inclusion in (i) and we omit it. This fininshes the proof.

Theorem 3.4.

(i) $T(S)$ *is special iff* S *is nonstationary in* ω_1.

(ii) $T(S) \otimes T(S')$ *is special iff* $S \cap S'$ *is nonstationary in* ω_1.

Proof. By theorems 2.4 and 3.3 we have: $T(S)$ is special iff $I_S = I_{T(S)} = \mathcal{P}(\omega_1)$ iff S is nonstationary. This proves (i).

For (ii) first notice that $T(S \cap S')$ is isomorphic to an initial part of $T(S) \otimes T(S')$. So by theorems 2.2 (ii), (iv) and 3.3 we have $\langle I_S \cup I_{S'} \subseteq I_{T(S)\otimes T(S')} \subseteq I_{S\cap S'}$. So $I_{T(S)\otimes T(S')} = I_{S\cap S'}$ since trivialy $\langle I_{S'} \cup I_{S'} \rangle = I_{S\cap S'}$. Now by theorem 2.4 we have: $T(S) \otimes T(S')$ is special iff $I_{S\cap S'} = I_{T(S)\otimes T(S')} = \mathcal{P}(\omega_1)$ iff $S \cap S'$ is nonstationary.

Remark 3.5 (1) In the proof of theorem 3.3 we have used a Skolem type argument so that using the technique of elementary submodels makes the proof a little shorter.

(2) In theorem 3.4 (ii) we could have a somewhat stronger result: for countable F, $\otimes\{T(S_i) \mid i \in F\}$ is special *iff* $\cap\{S_i \mid i \in F\}$ is nonstationary in ω_1.

(3) If S and S' are two disjoint stationary subsets of ω_1 then by Corollary 3.4, $T(S)$ and $T(S')$ are not special trees while $T(S) \otimes T(S')$ is special.

Definition 3.6. A tree T is *Baire* is for every sequence $\langle U_n \mid n < \omega \rangle$ of dense final parts of T, $\cap\{U_n \mid n < \omega\}$ is also dense in T.

Theorem 3.7 *For every uncountable S and S':*

(i) *$T(S)$ is Baire iff S is stationary in ω_1;*

(ii) *$T(S) \otimes T(S')$ is Baire iff $S \cap S'$ is stationary in ω_1.*

Proof: (i) Direct implication follows from theorem 3.4 (i) since if $T(S)$ is special then it is not Baire. Reverse implication appeared in the literature (see e.g. [BHK], [D]) but for the convenience of the reader let us sketch the proof of it here. Let $\langle U_n \mid n < \omega \rangle$ be a sequence of dense final parts of $T(S)$. Again we construct sequences $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ and $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$ as in the proof of theorem 2.2 (i), but now $u(t,s,i)$ has the properties $t \leq u(t,s,i)$, $\max(u(t,s,i)) > \delta_\alpha$ and $u(t,s,i) \in U_i$ (in the main case). Let $\delta_\alpha = \alpha \in S$ (exists since S is stationary). Then we can construct an increasing sequence $\langle t_n \mid n < \omega \rangle$ of elemenss A_α such that $\{\max(t_n) \mid n < \omega\}$ is unbounded in α and $t_n \in U_n$ for every n. Let $t = (\cup\{t_n \mid n < \omega\}) \cup \{\alpha\}$. Then $t \in T(S)$ and $t \in \cap\{U_n \mid n < \omega\}$. Since we can start from every element t_0 of $T(S)$ we are done.

(ii) Direct implication follows from theorem 3.4 (i) since any special (normal) tree is not Baire. For the reverse implication we have only to repeat the argument from the proof of reverse implication in (i).

Again in 3.7 (ii) we can have a somewhat stronger result: for countable F, $\otimes\{T(S_i) \mid i \in F\}$ is Baire iff $\cap\{S_i \mid i \in F\}$ is stationary in ω_1.

4. Continuums

Definition 4.1. Let S be a nonempty subset of ω_1 with no last element. Then by $\Gamma(S)$ we denote the set of all branches of the tree $T(S)$ (defined in §3). Let $\lhd$ be a linear order of S such that $(S, \lhd)$ is isomorphic to a dense subset of real numbers and such that for every $\alpha \in S$, $(\alpha, \alpha_\omega) \cap S$ is dense in $(S, \lhd)$ where α_ω is the $\omega\, th$ element of S above α. Let $\prec$ be the lexicographical order of $\Gamma(S)$ induced by $\lhd$, i.e. $b \prec b'$ iff $\max(t) \lhd \max(t')$ where $t = \min(b - b')$ and $t' = \min(b' - b)$. Let $(C(S), \prec')$ be the Dedekind completion of the linearly ordered set $(\Gamma(S), \prec)$. Then $(C(S), \prec')$ is a linearly ordered continuum since $(\Gamma(S), \prec)$ is clearly dense. We shall omit writing $\prec$ and $\prec'$ since $\prec$ and $\prec'$ are the only orderings on $\Gamma(S)$ and $C(S)$, resp. considered in this paper. Also we shall assume that $\Gamma(S)$ and $C(S)$ are linearly ordered topological spaces with the topology induced by these orderings, respectively.

Let $t \in T(S)$. Then by $B(t)$ we denote the set $\{b \in \Gamma(S) \mid b \ni t\}$. It is easy to see that $B(t)$ is an open convex subset of $\Gamma(S)$ and that $B(t)$, $t \in T(S)$ form a base of $\Gamma(S)$. Let $B'(t)$ be the convex closure of the set $B(t)$ in $C(S)$. Then $B'(t)$ is an open set in $C(S)$ and $B'(t)$, $t \in T(S)$ form a π-basis of $C(S)$.

In this section we are going to list some properties of the function $S \to C(S)$. S is always a nonempty subset of ω_1 such that $\lim(\mathrm{tp}\,(S))$.

Theorem 4.2. *The following propositions are eguivalent:*

(i) S *is stationary in* ω_1*;*

(ii) $C(S)$ *is not a* K_0*-space;*

(iii) $C(S)$ *has no* σ*-disjoint* π*-base;*

(iv) $C(S)$ *has no dense metrizable subspace.*

Proof: (ii)$\rightarrow$(iii)$\rightarrow$(iv) travially hold for every regular space. Let us prove (iv)$\rightarrow$(i). Assume (i), i.e. that S is nonstationary then by theorem 3.4 (i), $T(S)$ is special i.e. $T(S) = \cup\{A_n \mid n < \omega\}$ and A_n is maximal antichain of $T(S)$ for every $n < \omega$. Let $\mathcal{P}_n = \{B'(t) \mid t \in A_n\}$, then $\mathcal{P}_n$ is dicjoint familly of open subsets of $C(S)$, $\cup\mathcal{P}_n$ is dense in $C(S)$ and $\mathcal{P} = \cup\{\mathcal{P}_n \mid n < \omega\}$ is a π-basis of $C(S)$. Let $D = \cap\{\cup\mathcal{P}_n \mid n < \omega\}$ then D is dense metrizable subspace of $C(S)$ since $\{B'(t) \cap D \mid t \in A_n,\ n < \omega\}$ is a σ-discrete basis of D.

(i)$\rightarrow$(ii). Suppose $\neg$(ii) i.e., that there exists a dense in $C(S)$ set $D = \cup\{D_n \mid n < \omega\}$ such that D_n is discrete subspace for every $n < \omega$. Let $U_n = \{t \in T(S) \mid B'(t) \cap D_n = \emptyset\}$ then it is easy to see that U_n is a dense final part of $T(S)$ for every $n < \omega$. Since D is dense in $C(S)$ must be $\cap\{U_n \mid n < \omega\} = \emptyset$. So $T(S)$ is not Baire, and by theorem 2.7 (i) S not stationary in ω_1. This finishes the proof.

Remark 4.3. From theorem 4.2. we can conclude: If S is stationary in ω_1 then the union of countably many nowhere dense subsets of $C(S)$ is also nowhere dense.

Theorem 4.4. *The following are eguivalent for every uncountable* S*:*

(i) S *is costationary in* ω_1*;*

(ii) $C(S)$ *is first countablee;*

(iii) $C(S)$ *is the union of* $\aleph_1$ *nowhere dense subsets.*

Proof: If S is not costationary then $T(S)$ contains an ω_1-branch and so $\Gamma(S)$ and then $C(S)$ contains an uncountable well ordered or conversely well ordered subset, hence $C(S)$ is not first countable. This proves (ii)$\rightarrow$(i). Suppose now $C(S)$ is not first countable i.e. that contains an uncountable well ordered or conversely well ordered subset, hence $\Gamma(S)$ also has such a subset. It is easily seen that then $T(S)$ must contain an ω_1-branch. So $I_S = T_{T(S)} = NS$ by theorems 2.2 (ii) and 3.2, hence S is not costationary. This proves (i)$\rightarrow$(ii).

(i)$\rightarrow$(iii). Assume S is costationary. Since $C(S)$ is first countable (by (i)$\rightarrow$(ii)) it is enough to prove that $\Gamma(S)$ is the union of an increasing ω_1-sequence of nowhere dense subsets. Define $N_\alpha = \{b \in \Gamma(S) \mid \cup b \subseteq \alpha\}$, then N_α is nowhere dense is $\Gamma(S)$ and $N_\alpha \subseteq N_\beta$ for every $\alpha < \beta < \omega_1$. Since S is costationary $T(S)$ contains no ω_1-branch; hence $\Gamma(S) = \cup\{N_\alpha \mid \alpha < \omega_1\}$

(iii)$\rightarrow$(i). Assume that S is not costationary, i.e. that there exists a closed unbounded $C \subseteq S$ and that N_α, $\alpha < \omega_1$ is a familly of nowhere dense subsets of $B(S)$. Using C and standard arguments it is easy to construct an ω_1-branch $b \in \Gamma(S)$ which omits each N_α, $\alpha < \omega_1$. This completes the proof.

Theorem 4.5. *The following are eguivalent for every uncountable S:*

(i) *S is a bistationary subset in ω_1;*

(ii) *$C(S)$ is first countable and is not a K_0-space;*

(iii) *$C(S)$ is first countable and has no σ-disjoint π-base;*

(iv) *$C(S)$ is first couuntable and has no dense metrizable subspace;*

(v) *$C(S)$ is first countable and is not a Blumberg space.*

Proof: (i)$\leftrightarrow$(ii)$\leftrightarrow$(iii)$\leftrightarrow$(iv) follow from theorems 4.2 and 4.4. Let us prove (i)$\rightarrow$(v). Assume S is stationary and costationary, then $C(S)$ is first countable, has not σ-disjoint π-base and is the union of $\aleph_1$ nowhere dense subsets (by (i)$\leftrightarrow$(iii) and theorem 4.4). This is enough to conclude that $C(S)$ is not Blumberg by [W; Theorem 2]. (The interested reader can easily find a function $f : C(S) \to R$ for which there is no dense $D \subseteq C(S)$ such that $f \mid D$ is continuous. Note that if $D \subseteq C(S)$ is dense and $f : D \to Y$ continuous, where Y is a metrizable space, then there exists a nonempty open set B of D such that $f \mid B$ is constant.) Assume now that $C(S)$ is first countable non Blumberg space. By theorem 4.4 S is costationary. If S is not stationary then by theorem 4.2 $C(S)$ has σ-disjoint π-base, hence it is Blumberg by [Wh 1: Prop 1.7], contradiction. This completes the proof.

Now we are going to consider the function $(S, S') \to C(S) \times C(S')$.

Theorem 4.6. *The following are eguivalent for bistationary S and S':*

(i) *$S \cap S'$ is nonstationary in ω_1;*

(ii) *$C(S) \times C(S')$ is a K_0-space;*

(iii) *$C(S) \times C(S')$ has σ-disjoint π-base;*

(iv) *$C(S) \times C(S')$ has dense metrizable subspace;*

(v) *$C(S) \times C(S')$ is a Blumberg space.*

Proof: (iv)$\rightarrow$(iii)$\rightarrow$(ii) travilly hold for every regular space.

(ii)$\rightarrow$(i). Let $D = \cup\{D_n \mid n < \omega\}$ be a dense subset of $C(S) \times C(S')$ such that each D_n is a discrete subspace. Define $U_n = \{(s,t) \in T(S) \otimes T(S') \mid B'(s) \times B'(t) \cap D_n = \emptyset\}$. Then U_n is a dense final part of $T(S) \otimes T(S')$ since $T(S)$ and $T(S')$ are normal ω_1-trees. $\cap\{U_n \mid n < \omega\}$ is empty, since D is dense in $C(S) \times C(S')$, hence $T(S) \times T(S')$ is not Baire, and by theorem 3.7 (ii), $S \cap S'$ is not stationary in ω_1.

(i)$\rightarrow$(iv). Assume $S \cap S'$ is not stationary. Then by theorem 3.4 (ii), $T(S) \otimes T(S')$ is special. So there exists a family A_n, $n < \omega$ of maximal antichains of $T(S) \otimes T(S')$ such that $T(S') \otimes T(S') = \cup\{A_n \mid n < \omega\}$. Let $\mathcal{P}_n = \{B'(s) \times B'(t) \mid (s,t) \in A_n\}$ for $n < \omega$. Then $\mathcal{P} = \cup\{\mathcal{P}_n \mid n < \omega\}$ is a σ-disjoint π-base of $C(S) \times C(S')$ and $\cup\mathcal{P}_n$ is dense in $C(S) \times C(S')$ for every $n < \omega$. Let $D = \cap\{\cup\mathcal{P}_n \mid n < \omega\}$. Since $C(S) \times C(S')$ is Baire, D is dense subset of $C(S) \times C(S')$. Let $D' = \{(x,y) \in D \mid (x,y)$ has countable character in $C(S) \times C(S')\}$. It is easy to check that D' is a dense Baire subspace of $C(S) \times C(S')$. By [Wh 2; Theorem 2.6] we know that D' (and then $C(S) \times C(S')$) contains a dense metrizable subspace.

(i)$\to$(v). If $S\cap S'$ is nonstationary then by (i)$\to$(iii), $C(S)\times C(S')$ has a σ-disjoint π-base. Hence $C(S)\times C(S')$ is a Blumberg space with a σ-disjoint π-base ([Wh 1; Prop. 1.7]).

(v)$\to$(i) Assume; now that $C(S)\times C(S')$ is a Blumberg space. By Theorem 4.4 $C(S)$ is the union of $\aleph_1$ nowhere dense subsets and so is $C(S)\times C(S')$. By [W; Th. 5] there exists a countable collection $\mathcal{B}=\{B_n \mid n<\omega\}$ of open subsets of $C(S)\times C(S')$ such that for all open $B\subseteq C(S)\times C(S')$ there exists $(x,y)\in B$ and $\mathcal{B}'\subseteq\mathcal{B}$ such that $(x,y)\in\cap\mathcal{B}'$ but $\cap\mathcal{B}'$ contains no nonempty open set. Let $U_n=\{(s,t)\in T(S)\otimes T(S')\mid B'(s)\times B'(t)\subseteq B_n$ or $B'(s)\times B'(t)\subseteq C(S)\times C(S')-B_n\}$; then U_n is a dense final part of $T(S)\otimes T(S')$ for every $n<\omega$. Suppose $(s,t)\in\cap\{U_n\mid n<\omega\}$. By the property of the family $\mathcal{B}$ there exists $(x,y)\in B'(s)\times B'(t)$ and $\mathcal{B}'\subseteq\mathcal{B}$ such that $(x,y)\in\cap\mathcal{B}'$ but $\cap\mathcal{B}'$ contains no nonempty open set. Thus is imposible since clearly $B'(s)\times B'(t)\subseteq\cap\mathcal{B}'$. So $\cap\{U_n\mid n<\omega\}$ is empty, hence $T(S)\otimes T(S')$ is not Baire and $S\cap S'$ is not stationary by theorem 3.7 (ii). This completes the proof.

REMARK 4.7. (1) For the implications (i)$\to$(ii)$\to$(iii)$\to$(iv) and (i)$\to$(v) of theorem 4.6 it is enough to assume, for S and S', only that $\gamma T(S)=\gamma T(S')=\omega_1$. If we want (v)$\to$(i) we have also to assume that S or S' is costationary.

(2) Kurepa in [K 2] defined the K_0-property of topological spaces and proved that Suslin continuum is not a K_0-space. In order to construct a "Suslin like" continuum he constructed in [K 3] a first countable continuum and asked whether it was a K_0-space or not. Unfortunately, his example has the K_0-property.

If S is a bistationary subset of ω_1 (such set exists, see [S]) then $C(S)$ is a first countable linearly ordered continuum which is not a K_0-space (theorem 4.5).

(3) Let $J(\lambda)$ denote the ordrer type $\operatorname{tp}(\sigma R,\prec)+1$ (see def 3.1 (1)), where $\prec$ on σR is defined by: $s\prec t$ iff $s<t$ or $\min(s-t)<_R\min(t-s)$.

In [R; Problema, p. 330] Ricabarra asked whether $J(\lambda)$ contains every order type φ with properties $\varphi\ngeq\omega_1,\omega_1{}^*$. The answer is negative since the order type of $C(S)$ for S bistationary is a counterexample. This can be shown using theorem 4.5 and the fact that every linearly ordered set which is similar to a subset of $(\sigma R,\prec)$ has a σ-disjoint π-base.

(4) In [EČ; 2.12. II] Efimov and Čertanov asked for the existence, in ZFC, of a Fréshet-Urysohn compact space containing no metrizable dense subspace. Also in [Č] Čertanov asked for an absolute example of a first countable compact space containing no metrizable dense subspace. Clearly such an example is $C(S)$ above.

(5) In [A 1; Question 4] Arhangel'skiĭ asked: Is it true, under Martin's axiom and the negation of the continuum hypothesis, that every compact space of countable tightness is a K_0 space? Does it has a σ-disjoint π-base?

$C(S)$ for S constationary is clearly counterexample according to the theorem 4.5.

(6) Compact spaces that lie in a $\sum_{\aleph_0}$-product of intervals are called Corson spaces. (If A is an index set then $\{x\in[0,1]^A\mid\{\alpha\in A\mid x_\alpha\neq 0\}$ is countable$\}$ with the Tychonov topology is a $\sum_{\aleph_0}$-product of intervals.)

In [EČ; 2.12. I] Efimov and Čertanov asked: Is it true that every Corson space contains a metrizable dense subspace? Arhangel'skiĭ (see [A]; p. 74 and 84, resp.]) also asked for an absolute construction of a Corson, space containing no metrizable dense subspace.

Let S be a bistationary subset of ω_1. In [Š, Th. 1] Šapirovskiĭ proved that every compact space of countable tightness can be mapped irreducibly onto a certain Corson space. Since $C(S)$ is first countable there exist a Corson space $X(S)$ and an irreducible mapp $f : C(S) \to X(S)$. Since $C(S)$ has no σ-disjoint π-base and since f is irreducible, $X(S)$ also has no σ-disjoint π-base. So $X(S)$ is example which answers above questions. One can prove that if S and S' are disjoint stationary sets then $X(S) \times X(S')$ has a dense metrizable subspace. It is interesing to note that $X(S)$ cannot be an ordered space by [EČ; Th. 2.9].

(7) Let X and Y be regular topological spaces: X and Y are $\mathcal{G}$-absolute whenever they have isomorphic regular open algebras. If X an Y are compact $\mathcal{G}$-absolutnees is equal to co-absolutnees.

In [Wi; Question (1)] S. W. Williams asked for an absolute example of a compact first countable space not $\mathcal{G}$-absolute with any linearly ordered space.

If S is a bistationary subset of ω_1 and if [0,1] is the interval of real numbers then $[0,1] \times C(S)$ is such an example. This can be proved using the fact that any linearly ordered space has a π-basis which is a tree and that $[0,1] \times C(S)$ has no π-basis which is a tree. Let us note that this implies that $[0,1] \times C(S)$ is an example (in ZFC) of a compact first countable space which is not compactification of any linearly ordered space.

The fact that $[0,1] \times C(S)$ is not co-absolute with any compact linearly ordered space was first pointed to us by G. I. Čertanov who also noticed that this solves some of his problems.

(8) In [A 2; p. 47 or 51), Arhangel'skiĭ asked: Is the K_0-property preserved under perfect maps?

The answer is negative, because if S and S' are disjoint stationary sets in ω_1 then $C(S)$ and $C(S')$ do not have the K_0-property (theorem 4.2) while $C(S) \times C(S')$ has the K_0-property (theorem 4.6).

(9) In [W], W. Weiss constructed, in ZFC, an example of compact non-Blumberg space. His example is the topological sum of a compact linearly ordered space and the Stone space of the measure algebra. So it is natural to ask for an absolute example of compact linearly ordered non-Blumberg space. This question was explicitly asked by D. Lutzer in [L]. Also in connection with this Weiss' example, H. E. White [Wh 3; Remark (4)] asked: In ZFC, is there a compact non-Blumberg Hausdorff space that is the union of $\leq 2^{\aleph_0}$ nowhere dense subsets?

If S is a bistationary subset of ω_1, then $C(S)$ is a first countable linearly ordered continuum which is the union of $\leq \aleph_1$, nowhere dense subsets and which is not a Blumberg space, i.e., there exists $f : C(S) \to R$ which is not continuous on any dense subset of $C(S)$ (see theorem 4.5).

(10) In [W; Question 4 and 5], Weiss asked: Is the Blumberg property preserved under continuous open surjections? If $X \times Y$ is a Blumperg space, must both X and Y be Blumberg?

The answers are negative, because if S and S' are disjoint stationary subsets of ω_1 then $C(S)$ and $C(S')$ are not Blumberg spaces (theorem 4.5) while $C(S) \times C(S')$ is a Blumberg space (theorem 4.6).

5. Rigid Continuums

In this section we shall see that S determines the isomorphism type of the continuum $C(S)$. Using this we shall construct many rigid first countable linearly ordered continuums.

THEOREM 5.1. *If S and S' are stationary subsets of ω_1 and if $f : C(S) \to C(S')$ is one-to-one and continuous then $S - S'$ is not stationary in ω_1.*

Proof: Assume the contrary i.e., that $S_0 = S - S'$ is stationary in ω_1. Define, by induction, sequences $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ and $\langle {A_\alpha}' \mid \alpha < \omega_1 \rangle$ of countable subsets of $T(S)$ and $T(S')$, respectively and a normal sequence $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$ of ordinals in ω_1. Let $A_0 = {A_0}' = \{\emptyset\}$ and $\delta_0 = 0$. If $\lim \alpha$ then $A_\alpha = \cup\{A_\beta \mid \beta < \alpha\}$, ${A_\alpha}' = \cup\{{A_\beta}' \mid \beta < \alpha\}$ and $\delta_\alpha = \sup\{\delta_\beta \mid \beta < \alpha\}$. Now suppose that A_α, ${A_\alpha}'$ and δ_α are defined. Let $s \in A_\alpha$ and $t \in {A_\alpha}'$ be such that $f''(B'(s)) \cap B'(t) \neq \emptyset$. Let $u(s,t) \in T(S)$ be such that $s \leq u(s,t)$, $\max(u(s,t)) > \delta_\alpha$ and $f''(B'(u(s,t))) \subseteq B'(t)$. Otherwise let $u(s,t) = s$. Let $A_{\alpha+1} = A_\alpha \cup \{u(s,t)) \mid s \in A_\alpha,\ t \in {A_\alpha}'\}$. Let now $s \in A_{\alpha+1}$, and $t \in {A_\alpha}'$ be such that $f''(B'(s)) \subseteq B'(t)$. Since $B'(s)$ has no σ-disjoint π-base (theorem 4.2) it is easy to see that there exists $v(s,t) \in T(S')$ such that $t \leq v(s,t)$, $\max(v(s,t)) > \delta_\alpha$ and $B'(v(s,t)) \cap f''(B'(s)) \neq \emptyset$. Let $A'_{\alpha+1} = {A_\alpha}' \cup \{v(s,t) \mid s \in A_{\alpha_0+1}, t \in {A_\alpha}', f''(B'(s)) \subseteq B'(t)\}$ and let $\delta_{\alpha+1} = \sup\{\max(t) \mid t \in A_{\alpha+1} \cup A'_{\alpha+1}\}$.

Since S_0 is stationary in ω_1 and $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$ normal, we can find $\alpha < \omega_1$, such that $\delta_\alpha = \alpha \in S_0$. Let $\langle \alpha_n \mid n < \omega \rangle$ be an increasing sequence cofinal in α. By induction we define an increasing sequences $\langle s_n \mid n < \omega \rangle$ and $\langle t_n \mid n < \omega \rangle$ of elements of A_α and ${A_\alpha}'$, resp. Let $s_0 = t_0 = \emptyset$. Since $f''(B'(s)) \cap B'(t_0) \neq \emptyset$ there is an $s_1 \in A_{\alpha_0+1} \subseteq A_\alpha$ such that $s_0 \leq s_1$, $\max(s_1) > \alpha_0$ and $f''(B'(s_1)) \subseteq B'(t_0)$ (by the definition of $A_{\alpha+1}$). By the definition of A'_{α_0+1} there exists a $t_1 \in A'_{\alpha_0+1} \subseteq {A_\alpha}'$ such that $t_0 \leq t_1$, $\max(t_1) > \alpha_0$ and $B'(t_1) \cap f''(B'(s_1)) \neq \emptyset$. Now there is an $s_2 \in A_\alpha$ such that $s_1 \leq s_2$, $\max(s_2) > \alpha_1$ and $f''(B'(s_1)) \subseteq B'(t_1)$ and then $t_2 \in {A_\alpha}'$ such that $t_1 \leq t_2$, $\max(t_2) > \alpha_1$ and $B'(t_2) \cap f''(B'(s_2)) \neq \emptyset$, etc. We proceed in this manner. Clearly $s = (\cup\{s_n \mid n < \omega\}) \cup \{\alpha\} \in T(S)$ and by the construction $f''(B'(s)) \subseteq \cap\{f''(B'(s_n)) \mid n < \omega\} \subseteq \cap\{B(t_n) \mid n < \omega\}$. But this is a contradiction since $|f''(B'(s))| > |\cap\{B'(t_n)|n < \omega\}| = 1$. This finishes the proof.

REMARK 5.2. (1) Let $S \subseteq \omega_1$. Then $C^*(S)$ denotes a linearly ordered compact zerodimensional space obtained from $C(S)$ by replacing every point $x \in C(S)$ by two new points x^- and x^+ (ordered naturally). If $t \in T(S)$ then $B^*(t)$ denotes the set $\{x^-, x^+ \mid x \in B'(t)\}$. Then $B^*(t)$, $t \in T(S)$ form a π-basis of $C^*(S)$. Repeating the same arguments as above we can prove:

(a) If S is stationary in ω_1 then $C^*(S)$ has no σ-disjoint π-base;

(b) If S is stationary in ω_1 then every first category subset of $C^*(S)$ is nowhere dense (see Remark 4.3);

(c) The theorem 5.1 holds for the function $S \to C^*(S)$.

(2) Using a family of $\aleph_1$ disjoint stationary subsets of ω_1 (see [S]) we can, in a standard way, construct a family $\mathcal{S}$ of $2^{\aleph_1}$ stationary subsets of ω_1 such that $S - S'$ is stationary for every $S, S' \in \mathcal{S}$, $S \neq S'$. So, the corresponding family $C(S)$, $S \in \mathcal{S}$ of first countable lindarly ordered continuums has the following properties:

(a) $C(S)$ is not a K_0-space for every $S \in \mathcal{S}$;

(b) If S, $S' \in \mathcal{S}$ are diferent, then $C(S)$ is not homeomorphic to a subspace of $C(S')$.

Theorem 5.3. *There is a family $\mathcal{C}$ of power $2^{2^{\aleph_0}}$ of first countable linearly ordered continuums such that:*

If C, $C' \in \mathcal{C}$ and if $f : C \to C'$ is continuous and one-to-one then $C = C'$ and $f = id_C$.

Proof: Let $\mathcal{F} \subseteq \mathcal{S}$ ($\mathcal{S}$ is the family of stationary sets from the remark 5.2. (2)) and let $|\mathcal{F}| = 2^{\aleph_0}$. Choose a decomposition $\mathcal{F} = \cup\{\mathcal{F}_n \mid n < \omega\}$ such that $\mathcal{F}_0 = \{S_0\}$, $|\mathcal{F}_n| = 2^{\aleph_0}$ for $n \geq 1$ and $\mathcal{F}_n \cap \mathcal{F}_m = \emptyset$ for $n \neq m$. By induction we define an increasing sequence $\langle C_n \mid n < \omega\rangle$ of lin. ordered continuums as follows. Let $C_0 = C(S_0)$. Suppose we have already constructed C_n. Choose a bijection $x \to S_x$ from C_n onto $\mathcal{F}_{n+1}$. Let C_{n+1} is obtained from C_n by replacing every point x of C_n with $C(S_x)$. Assume $C_n \subseteq C_{n+1}$ identifying $x \in C_n$ with $\min C(S_x)$. Let $C = C(\mathcal{F})$ be the Dedekind completion of $\cup\{C_n \mid n < \omega\}$. Clearly C is a first countable continuum.

Let us prove that if there exists a continuous and one-to-one mapping $f : C(\mathcal{F}) \to C(\mathcal{F}')$ then $\mathcal{F} \subseteq \mathcal{F}'$. Assume the contrary, i.e. $S \in \mathcal{F} - \mathcal{F}' \neq \emptyset$. Then $C(S)$ is a subset of $C(\mathcal{F})$. It is easy to see that $\overline{C(S)} \cong C^*(S)$ where $C^*(S)$ is space from Remark 5.2. (1). We shall identify $\overline{C(S)}$ and $C^*(S)$. Let us prove that there exists an $n < \omega$ such that $f''C^*(S) \cap C_n(\mathcal{F}')$ is not nowhere dense in $f''C^*(S)$. Otherwise by replacing $C^*(S)$ by $B^*(t)$ we can assume that $f''C^*(S) \cap C(\mathcal{F}') = \emptyset$ for every n (see Remark 5.2. (1) (b)). But then $f''C^*(S)$ (and then $C^*(S)$) has a σ-disjoint π-base as is easily seen; contradiction (see Remark 5.2 (1) (a)). So there exists $n = \min\{m \mid f''C^*(S) \cap C_m(\mathcal{F}')$ is not nowhere dense in $f''C^*(S)\}$. By replacing $C^*(S)$ by $B^*(t)$ (some $t \in T(S)$) we can assume $f''C^*(S) \subseteq C_n(\mathcal{F}')$. By the minimality of n, in case $n > 0$, we can assume w.l.o.g. that $f''C^*(S) \subseteq C(S_x)$ for some $x \in A_{n-1}(\mathcal{F}')$. Now we have a contradiction with Remark 5.2. (1) (c) since $S - S_x$ is stationary in ω_1.

Using the same orguments we can prove that every continuous one-to-one $f : C(\mathcal{F}) \to C(\mathcal{F})$ must be trivial. This finishes the proof since there exists a family of power $2^{2^{\aleph_0}}$ of subsets of $\mathcal{S}$ no one of which is included in any other.

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