PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 29 (43), 1981, pp. 269–282

ON SPECTRA OF INFINITE GRAPHS

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Dedicated to Prof. dr. Djuro Kurepa in honour of his
73. birthday (1980)

Abstract. We introduce the notion of the spectrum of an infinite denumerable graph, with specially defined adacency matrix. Among other things, we investigate general properties of spectra, spectra of bipartite graphs and infinite graphs with finite spectrum.

The main difference in comparision with spectra of finite graphs is the non-uniqueness of the spectrum of an infinite graph.

1.Introduction.

1. – There were few attempts to extend the well developed theory of spectra of finite graphs [5] to the infinite case (see, for example [2*a*]). Here and in the forthcoming papers we offer such an extension.

Thus our aim is to introduce and to investigate the spectrum of an infinite graph, and to obtain an analogue to the spectral theory of finite graphs. But this idea comes accross distinct difficulties, and only a number of results rest valid for infinite graphs.

In this paper, we adopt without a special mention many general denotations and definitions from the spectral theory of finite graphs (for instance, bipartite graphs, complete graphs, irreducible matrix, prenumeration of the vertex set i.e. permutation of a graph etc).

Throughout the paper, under *a graph* G we always mean an *infinite denumerable graph* with vertex set $V(G)$ equal to the set $N = \{1, 2, \dots\}$ of natural numbers, which is in addition *connected* and *undirected*, and without loops or multiple edges.

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1) – Communicated on VII Congress of Yugoslav Mathematicians, 6–12. 10. 1980, Bečići–Budva.

2) – Izradu ovog rada je finansirala Republička zajednica za naučni rad SR Srbije.

The adjacency matrix $A(G) = [a_{ij}]$ of G is an infinite $N \times N$ matrih, which is, in order to avoid several difficulties, defined in a special way; we put

$$a_{ij} = \begin{cases} a^{i+j-2}, & \text{if vertices } i \text{ and } j \text{ are adjacent,} \\ 0, & \text{if } i \text{ and } j \text{ are non-adjacent} \end{cases}$$

where a is a fixed positive constant $(0 < a < 1)$.

Next, since if i, j are adjacent, $a_{ij} = a^{i-1}a^{j-1}$, we can say that for any $i \in N$ vertex v_i supports the weight a^{i-1}, so that whole graph G is labelled or weighted.

As an essential difference in comparision with finite case, we note that the matrix $A = A(G)$, at any relabelling of the vertex set $V(G)$ transforms into a matrix PAP', but the matrix P is never orthogonal (except in the trivial case).

Matrix A can be regarded as the matrix of a linear operator in a separable Hilbert space H with an orthonormal basis $\{e_1, e_2, \dots\} = \{e_i\}$ and it is obviously symmetric. In the sequel, we shall not differ it from the corresponding linear operator in space H.

We notice that in general case, spectrum of any bounded operator (or bounded symmetric operator) in a Hilbert space, does not consists of the eigenvalues only, so we need some auxilliary facts concerning symmetric Hilbert– Schmidt operator whose matrix has non-negative entries.

2. – Let $A = [a_{ij}]$ be any symmetric $N \times N$ matrix with complex entries such that

$$n(A) = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2} < \infty.$$

Then A is said to have finite absolute norm $n(A)$, it is bounded, and its operator norm $\|A\| \le n(A)$.

The corresponding operator A is also called – Hilbert-Schmidt operator. It is compact and self adjoint ([1], p.92), and its spectrum is the spectrum of the matrix A.

Since A is compact and self-adjoint in H, its spectrum is real and consists of a sequence $\lambda_1, \lambda_2, \lambda_3, \dots$ of eigenvalues (each of finite multiplicity), and of the value $\lambda = 0$ (which need not be an eigenvalue:

$$\sigma(A) = \{\lambda_1, \lambda_2, \dots\} \quad (|\lambda_1| \ge |\lambda_2| \ge \dots).$$

Here, sequence $\{\lambda_i\}$ is finite or $\lambda_n \to 0 (n \to \infty)$. The spectral radius $r = r(A)$ is equal to the maximal eigenvalue (and to the operator norm), $r(A) = \lambda_1 = \|A\|$.

Since $|\lambda_n| \le |\lambda_1|$, the whole spectrum $\sigma(A)$ is concentrated in the interval $[-r(A), r(A)]$.

If $\lambda = \lambda_i$ is an eigenvalue of A, then $N(A - \lambda I) = \{x \in H | Ax = \lambda x\}$ is the corresponding proper subspace of the operator A.

If now $\{f_v\}(v \in \Lambda)$ is any subset of vectors in H, then $\bar{L} = \bar{L}\{f_v \mid v \in \Lambda\}$ is the corresponding closed subspace of H generated by vectors $f_v \in H$.

Next if x is an arbitrary vector in space $H, x = \sum x_i e_i$, i.e. $x = (x_1, x_2, \dots)^\top$, it is said to be positive (non-negative), or negative (non-positive), if all its coordinates $x_i (i = 1, 2, \dots)$ are of the corresponding kind. Its norm is $\|x\| = (\sum |x_i|^2)^{1/2}$, and then

$$x^+ = \sum_{i=1}^{\infty} |x_i| e_i, \qquad x^- = -x^+ = -\sum_{i=1}^{\infty} |x_i| e_i.$$

We note that $\|x^+\| = \|x^-\| = \|x\|$.

As in the finite case, a Hilbert-Schmidt operator A is called "irreducible" (in the matrix sence), if there is no any coordinate space $\pi = \bar{L}\{e_{i_1}, e_{i_2}, \dots\}$, which is invariant for A.

It is easy to see that the adjacency matrix $A = A(G)$ of an infinite graph G is an irreducible operator iff the corresponding graph G is connected.

In the following theorem we quote some known facts from the theory of irreducible Hilbert-Schmidt operators with non-negative entries (see [8] and [9]).

THEOREM 1. – *Let $A = [a_{ij}]$ be an irreducible Hilbert-Schmidt symmetric operator with non-negative entries.*

Then $\lambda_1 = \|A\|$ is (maximal) eigenvalue of A, and it is simple. There is at least one positive eigenvector corresponding to this eigenvalue $\lambda_1 = r(A)$.

If $\lambda = -r(A)$ is the (minimal) eigenvalue of A, then it is is simple also. □

2. Spectra of infinite graphs.

1. – Let G be any infinite graph with adjacency matrix A. Then

$$n(A)^2 = \sum_{i,j} a_{ij}^2 \le \sum_{i \ne j} a^{i+j-1} = n(A_0)^2 < \infty,$$

where $n(A_0) = a\sqrt{2}/(1-a^2)\sqrt{1+a^2}$ is the absolute norm of the adjacency matrix A_0 of complete graph K_∞, thus A is a Hilbert-Schmid operator in the space H.

Then, spectrum $\sigma(G)$ of G is defined by $\sigma(G) = \sigma(A)$, thus as the spectrum of the corresponding Hilbert-Schmidt operator $A = A(G)$.

Applying Theorem 1, we immediately have following basic properties of the spectrum of an infinite graph G.

THEOREM 2. *–Spectrum $\sigma(G)$ of any infinite graph G consists from a sequence of real eigenvalues $\lambda_1 = r(G), \lambda_2, \lambda_3, \dots,$ and of zero ($\lambda_n \to 0, n \to \infty$), each non-zero λ_n has a finite multiplicity.*

The spectrum

$$\sigma(G) = \{\lambda_1, \lambda_2, \dots ; 0\}$$

is concentrated in the interval $[-r, r]$, *where* $r = r(G)$ *is the index (spectral radius of* G.

The maximal eigenvalue $\lambda = r(G)$ *is simple.*

If $\lambda = -r(G)$ *is the (minimal) eigenvalue of* G, *it is simple too.* $\square$

In such a way, we conclude that spectra of all considered infinite graphs lie in the interval $D = [-b, +b]$, where $b = a\sqrt{2}/(1-a^2)\sqrt{1+a^2}$. It is an open question to small this interval as much as possible.

Remark. – It is obviously that the spectrum $\sigma(G)$ of G depends on the constant $a(0 < a < 1)$ and of the way of its vertices. In a general case, it changes under relabelling of the vertex set $V(G) = N$.

But in spite of these difficulties, there is some number of spectral properties of the graph G which does not depend on a constant a and of the way of labelling (for instance, the property – to have the finite spectrum). It is naturally to call them – *pure spectral properties* of graph G. Hence, to have finite spectrum (or to have infinite spectrum) – is a pure spectral property of G.

It is obviously of great interest to find as many as possible such (pure spectral) properties of graph G.

2. – Consider especially relabelling (prenumeration) of the vertex set VG of a graph G, which corresponds to an arbitrary permutation $\omega = (\omega(1), \omega(2), \dots)$ of the set N; let the new graph so obtained be G_1.

If A_1 is the adjacency matrix of the graph G_1, let define a matrix $P = [p_{ij}]$ (permutation matrix) by

$$p_{ij} = \begin{cases} a^{i-\omega(i)}, & j = \omega(i) \\ 0, & j \neq \omega(i) \end{cases};$$

then $A_1 = PAP'$.

It can be easily checked that

$$PP' = \operatorname{diag}(a^{2-2\omega(1)}, a^{4-2\omega(2)}, \dots),$$
$$P'P = \operatorname{diag}(a^{2\omega^{-1}(1)-2}, a^{2\omega^{-1}(2)-4}, \dots),$$

so that matrix P is never orthogonal (except in the trivial case).

It is the reason that we lose the property $\sigma(G_1) = \sigma(G)$, property of invariatneess of spectrum at any prenumeration of the vertex set. $\square$

3. – Now add a new adge to the graph G; let the new graph be G_1 (we note that the vertex set VG_1 is assumed to be indexed in the same way as VG).

Proposition 1. – *Spectral radius* $r(G) < r(G_1)$.

Proof. – Let $A = [a_{ij}]$, $B = [b_{ij}]$ be adjacency matrices of G and G_1 respectively. Then $a_{ij} \leq b_{ij} (i, j \in N)$, and there is just one pair $i_0, j_0 (i_0 < j_0)$ such that

$$a_{i_0 j_0} = 0 < b_{i_0 j_0} = a^{i_0+j_0-2}.$$

If $x(\|x\|=1)$ is a proper vector of A corresponding to the eigenvalue $r(G)=\|A\|$, we can suppose it is positive, so that

$$r(G)=\langle Ax,x\rangle=\sum_{i,j}a_{ij}x_ix_j<\sum_{i,j}b_{ij}x_ix_j=$$
$$=\langle Bx,x\rangle\leq\|B\|=r(G_1),\quad \text{q.e.d.}\square$$

COROLLARY. – *If $r_0=r(G_0)$ is the spectral radius of complete graph $G_0=K_\infty$, then the spectrum $\sigma(G)$ of every graph G lies in the intervale $(-r_0,r_0)$.* $\square$

Moreover we note that for complete graph K_∞, it holds the strong inequlity $r_0=\|A_0\|<n(A_0)$. Indeed, equality $\|A\|=n(A)$ holds for an symmetric Hilbert-Schmidt matrix A, iff the operator A is one dimensional, which is one dimensional, which is impossible for any adjacency matrix A.

3. Spectrum of the complete graph K_∞.

THEOREM 3. – *Spectrum of the complete graph $G=K_\infty$ consists of an infinite seguence of simple eigenvalues $\xi_0,\xi_1,\xi_2,\dots$ where*

$$\xi_0>0,\quad \xi_1<\xi_2<\xi_3<\ldots<0.$$

It holds:

1) $\quad\xi_0>a^2;$

2) $\quad\xi_n\in(-a^{2n-2},-a^{2n})\quad(n=1,2,\dots),$

3) $\quad\xi_0=|\xi_1|+|\xi_2|+\dots$

PROOF. – Let λ be any eigenvalue of $G, x=(x_1,x_2,\dots)^\top\neq 0$ be a corresponding eigenvector. Then from $Ax=\lambda x$, we find:

$$(*)\qquad \begin{cases} f=x_1+ax_2+a^2x_3+\cdots=(\lambda+1)x_1\\ a^nf=(\lambda+a^{2n})x_{n+1}\quad(n=1,2,\dots)\end{cases}$$

Using condition

$$\|x\|^2=\sum_{i=1}^{\infty}x_i^2<\infty,$$

it can be eecily proved that $\lambda\neq 0$ and $\lambda\neq -a^{2n-2}(n=1,2,\dots)$.

So we obtain:

$$(1)\qquad x_n=(\lambda+1)a^{n-1}x_1/(\lambda+a^{2n-2})\qquad(\geq 2).$$

Substituting x_n in the first relation in $(*)$, we get

$$(\lambda+1)x_1 \sum_{n=1}^{\infty} a^{2n}/(\lambda+a^{2n}) = \lambda x_1.$$

Since $x_1 \neq 0$, we conclude that all eigenvalues of G satisfy the equation

$$\boxed{g(\lambda) = \sum_{n=1}^{\infty} a^{2n}/(\lambda+a^{2n}) = \lambda/(\lambda+1).} \tag{2}$$

We observe that, all roots of (2) are eigenvalues of G because

$$x_n^2 = \frac{(\lambda+1)^2 a^{2n-2} x_1^2}{(\lambda+a^{2n-2})^2} \sim (1+1/\lambda)x_1^2 a^{2n-2} (|a|<1),$$

so the series $\sum x_i^2$ converges.

Next, since it can be checked that $g'(\lambda) < 0$ in all intervals $(-\infty,-1)$, $(-1,-a^2),(-a^2,-a^4),\dots;(0,+\infty)$, the function $g(\lambda)$ is strongly monotonically decreasing in all these intervals.

On the other side, function $h(\lambda) = \lambda/(\lambda+1)$ is strongly monotonically incressing in intervals $(-\infty,-1)$, $(-1,+\infty)$, so we easily conclude that equation (2) has exactely one root in each of the intervals $I_n = (-a^{2n-2}, -a^{2n})$, $I_0 = (0,+\infty)$. Moreover, since the cooresponding proper vector is always unique defined, all these eigenvalues are simple.

Next, since $g(a^2) > h(a^2)$, we get that $\xi_0 > a^2$.

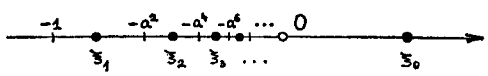

Fig. 1

To prove the equality

$$\xi_0 = \sum_{n=1}^{\infty} |\xi_n|,$$

let us first observe that in view of the estimation $\|\xi_n\| \le a^{2n-2} (n=1,\dots)$, the series

$$o\sum_{n=0}^{\infty} |\xi_n| < \infty,$$

converges.

But this is the characteristic property of so called *nuclear* operators, whose class is denoted by $\mathcal{L}_1(H)$ ([7], p. 88).

But then, spectral trace of A coincides to its matrix trace ([7], p. 127); i.e.

$$\sum_{n=0}^{\infty} \xi_n = \sum_{n=1}^{\infty} a_{nn} = 0,$$

thus

$$\xi_0 = \sum_{n=1}^{\infty} |\xi_n|, \quad \text{q.e.d.} \ \square$$

Problem. – Does for every infinite graph G, the adjacency matrix A is a nuclear operator, and consequently equality

$$\sum_{n=0}^{\infty} \xi_n = 0$$

holds. $\square$

4. Spectra of bipartite graphs.

THEOREM 4. – *Spectrum of any (connected) bipartite graph G is symmetric with respect to zero.*

PROOF. – Let the set $N = VG$ of vertices be devided into two subsets $N_1 = VG_1, N_2 = VG_2$, where N_1, N_2 are internel stabil:

$$a_{ij} = 0 \quad (i, j \in N_1, \quad \text{or} \quad i, j \in N_2).$$

If we put: $H_1 = \bar{L}\{e_i \mid i \in N_1\}$, $H_2 = \bar{L}\{e_j \mid j \in N_2\}$, then H_1, H_2 are closed mutually orthogonal subspaces of the space H, and $H = H_1 \oplus N_2$.

Then the operator A in the permutated basis $\{e_i \mid i \in N_1\} \cup \{e_j \mid j \in N_2\}$ has the form

$$A = \begin{bmatrix} 0 & B' \\ B & 0 \end{bmatrix}.$$

Now $Ax = \lambda x (\lambda \in R, x \neq 0)$, where $x = x_1 + x_2 (x_1 \in H_1,\ x_2 \in H_2)$ is equivalent to

$$\begin{bmatrix} 0 & B' \\ B & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix},$$

or to:

(1) $$\begin{cases} Bx_1 = \lambda x_2 \\ B' x_2 = \lambda x_1 \end{cases}.$$

Suppose that $\lambda \neq 0$ and observe that (1) implies $x_1 \neq 0$, $x_2 \neq 0$.

Then from (1),

$$B'Bx_1 = \lambda^2 x_1 \qquad (x_1 \neq 0). \tag{2}$$

Thus if $\lambda \neq 0$ is an eigenvalue of A, relation (2) must be satisfied.

Conversely, if $\mu = \lambda^2 > 0$ is any eigenvalue of $B'B \in B(H_1)$ whose corresponding eigenvector is x_1, then defining

$$x' = x_1 + Bx_1/\sqrt{\mu},$$
$$x'' = x_1 - Bx_1/\sqrt{\mu},$$

one obtain the mutually orthogonal eigenvectors of A corresponding to eigenvalues $\sqrt{\mu}$ and $-\sqrt{\mu}$ respectively.

We remark that $B'B$ is a non-negative Hilbert-Schmidt operator in the subspace H_1, so its spectrum $\sigma(B'B)$ consists from a sequence $\mu_1, \mu_2, \dots$ of real eigenvalues such that $\mu_n \to 0 (n \to \infty)$, including zero (if H_1 is infinite dimensional).

Hence the spectrum of G is quite described:

$$\sigma(A)\backslash\{0\} = \{\pm\lambda \mid \lambda^2 \in \sigma(B'B)\backslash(0)\}.\square$$

– We do not know does the converse true, i.e. is an infinite graph G bipartite if its spectrum is symmetric about zero.

We conjecture – no, but we have not any counter-example until now.

5. Spectrum of complete bipartite graph $K(N_1, N_2)$.

Let $G = K(N_1, N_2)$ be complete bipartite graph, where

$$N_1 = \{i_1, i_2, \dots\}, \quad N_2 = \{j_1, j_2, \dots\}$$

(some of N_1, N_2 can be finite too).

THEOREM 5. – *Spectrum $\sigma(G)$ is finite,*

$$\sigma(G) = \{0, \pm r\},$$

where

$$r = \sqrt{\sum_p a^{2i_p-2}} \sqrt{\sum_q a^{2j_q-2}}.$$

PROOF. – We have $a_{ij} = 0$ if $i = j$, or if i, $j \in N_1$, or if i, $j \in N_2$, and

$$a_{ij} = a^{i+j-2}, \qquad \text{otherwise.}$$

If λ is any eigenvalue of G, and $x \in N(A-I)(x \neq 0)$, we find from $Ax = \lambda x$:

$$\text{(1)} \qquad \begin{cases} \sum_q a_{i_p j_q} x_{j_q} = \lambda x_{i_p} & (i_p \in N_1) \\ \sum_p a_{j_r i_s} x_{i_s} = \lambda x_{j_r} & (j_r \in N_2) \end{cases}$$

Hence:

$$\text{(2)} \qquad a^{i_p-2} \sum_q a^{j_q} x_{j_q} = \lambda x_{i_p} \quad (i_p \in N_1)$$

$$\text{(3)} \qquad a^{j_r-2} \sum_s a^{i_s} x_{i_s} = \lambda x_{j_r} \quad (j_r \in N_2).$$

If now H_1, H_2 have the same meaning as in the preceeding section, and $x = x' + x''(x' \in H_1, x'' \in H_2)$, we easily find for $\lambda = 0$:

$$\sum_s a^{i_s} x_{i_s} = 0, \qquad \sum_q a^{j_q} x_{j_q} = 0,$$

so that x' is orthogonal to the vector $a' = (a^{i_1}, a^{i_2}, \dots)$ in H_1 and x'' is orthogonal (in H_2) to the vector $a'' = (a^{j_1}, a^{j_2}, \dots)$. Hence, x is an arbitrary vector from the orthogonal complement $\bar{L}\{f_1, f_2\}^{\perp}$, where $f_1 = \sum_i b_i e_i$, $f_2 = \sum_i c_i e_i$ and

$$\begin{cases} b_{i_p} \equiv a^{i_p}, & b_{j_q} \equiv 0, \\ c_{i_p} \equiv 0, & c_{j_q} \equiv a^{j_q}. \end{cases}$$

Thus $\lambda = 0$ is an eigenvalue, and the corresponding proper subspace $N(A)$ is of codimension 2.

If next $\lambda \neq 0$, then substituting x_{j_r} from (3) to (2), we find:

$$\frac{1}{\lambda^2} a^{i_p-2} (\sum_q a^{2j_q-2})(\sum_s a^{i_s} x_{i_s}) = x_{i_p} \qquad (i_p \in N_1).$$

Where from we obtain

$$\lambda^2 = \sum_p a^{2i_p-2} \sum_q a^{2j_q-2},$$

or $\lambda_{1,2,} = \pm r$, where

$$r = \sqrt{\sum_p a^{2i_p-2}} \sqrt{\sum_q a^{2j_q-2}} = \sqrt{l}\sqrt{1/(1-a^2) - l},$$

$$l = \sum a^{2i_p-2}.$$

For these values of λ, corresponding eigenvectors are for example:

$$x = \sum_p x_{i_p} e_{i_p} + \sum_q x_{j_q} e_{j_q}, \quad \text{where}$$
$$x_{i_p} = a^{i_p - 2} / (\sum_q a^{2j_q - 2}),$$
$$x_{j_q} = a^{j_q - 2} / \lambda.$$

Hence, spectum $\sigma(G) = \{0, \pm r\}$, q.e.d. □

If specially, $N_1 = \{i\}$, we get that $\sigma(G) = \{0, \pm r\}$, where

$$r = a^{i-1} \sqrt{1/(1-a^2) - a^{2i-2}}. \square$$

REMARK. – It can be proved that for distinct partitions N_1, N_2, all the graphs $G = K(N_1, N_2)$ have different spectra.

We do not know too, any non trivial example of two cospectral graphs, or of any graph G whose some non-zero eigenvalue is not simple. □

6. Line graph of an infinite graph.

Let G be an infinite graph. Then as in finite case, the line graph $L(G)$ of G is the graph whose vertices are edges of G[1], with two vertices of $L(G)$ – adjacent if and only if the corresponding edges of G have one vertex in common.

If now $g_i = (v_{p(i)}, v_{q(i)})(p(i) < g(i); i = 1, 2, \dots)$, we introduce the vertex-edge incidence matrix $R = [r_{ij}]$ of G defining

$$r_{ij} = \begin{cases} a^{j-1}, & i = p(j), q(j) \\ 0, & \text{otherwise} \end{cases}$$

Then:

$$\begin{aligned} Re_i &= r_{p(i)i} e_{p(i)} + r_{q(i)i} e_{q(i)} = \\ &= a^{i-1} [e_{p(i)} + e_{q(i)}] \qquad (i = 1, 2, \dots). \end{aligned}$$

If now A and B are adjacency matrices for G and $L(G)$ respectively, one can derive the basic formula

$$\boxed{R'R = B + 2D}, \tag{1}$$

where $D = \operatorname{diag}(1, a^2, a^4, \dots)$ ([4], p. 103).

REMARK. – For regular graphs with finite degree $d = d(G)$, in general we lose relation

$$RR' = A + dI.$$

[1] For our aim, it is not important the way of denumeration of vertices in $L(G)$.

It is the reason that we lose satisfactory results for *total graph* of a graph. □

We observe that the operator $R \in B(H)$ is not symmetric, but it is of finite absolute norm because

$$\sum_{i,j} r_{ij}^2 = 2\sum_{j=1}^{\infty} a^{2j-2} = 2/(1-a^2) < \infty.$$

Hence, $R'R$ is *a* Hilbert Schmidt operator too. □

THEOREM 6. – *For every infinite graph G, all spectral values $\lambda \in \sigma(L(G))$ satisfy the ineguality $\lambda > -2$.*

PROOF. – We have from (1),

$$B = A(L(G)) = R'R - 2\,\mathrm{diag}\,(1, a^2, a^4, \dots),$$
$$B + 2I = R'R + 2\,\mathrm{diag}\,(0, 1-a^2, 1-a^4, \dots) \geq 0,$$

because $R'R \geq 0$, and

$$\mathrm{diag}\,(0, 1-a^2, 1-a^4, \dots) \geq 0.$$

Hence $\sigma(B+2I) = \sigma(B) + 2 \geq 0$, so that $\lambda \geq -2$ for every $\lambda \in \sigma(B)$.

We yet prove that $\lambda = -2$ is not an eigenvalue of $B = A(L(G))$.

If oppositely, x is an eigenvector corresponding to $\lambda = -2$, we would have

$$< R'Rx, x >= 0, \quad \|Rx\|^2 = 0, \quad Rx = 0,$$

ane $Dx = 0$, which implies that $x = x_1 e_1 (x_1 \neq 0)$.

But then:

$$Rx = x_1 Re_1 = x_1[r_{p(1)1} e_{p(1)} + r_{q(1)1} e_{q(1)}] = 0,$$

which is impossible.

In this way, we have obtained the strong estimation $\lambda > -2$ for every $\lambda \in \sigma(B)$. □

A problem. – Does for every $\varepsilon > 0$, there exists an infinite graph G such that its minimal eigenvalue $\lambda_m(G) \in (-2, -2+\varepsilon)$. □

7. Graphs with finite spectra.

It is an important question – when an infinite graph G has the finite spectrum.

PROPOSITION 2. – *Spectrum $\sigma(A)$ of a compact selfadjoint operator $A \in B(H)$ is finite,*

$$\sigma(A) = \{\lambda_1, \dots, \lambda_p, 0\} \quad (\lambda_i \neq 0)$$

if and only if its range $R(A)$ is finite dimension p.

Then $\lambda = 0$ is an eigenvalue of A such that the corresponding proper subspace $N(A)$ is of codimension p. □

Here $\lambda_1, \dots, \lambda_p$ can be distinct or not, but their total number is p.

Next, since columns $C_1, C_2, \dots$ of the adjacency matrix A are in accordance with vectors $Ae_1, Ae_2, \dots$ in H, we get that spectrum $\sigma(A)$ is finite if and only if the adjacency matrix A has just p linearly independent columns.

Hence, $\sigma(A)$ contains p non-zero eigenvalues iff A has exactely p linearly independent columns.

Wherefrom we obtain the following.

Proposition 3. – *Spectrum $\sigma(G)$ of a graph G consists of exactly p non-zero eigenvalues (and zero), iff there at Ieast one minor of the adjacency matrix of order p different from zero, and all other minors of order $p+1$ are egual to* 0. □

Furher, if spectrum $\sigma(G)$ of a graph G is finite, and consists of p non-zero eigenvalues $\lambda_1, \dots, \lambda_p$, the operator A is obviously nuclear, i.e.

$$\sum_i |\lambda_i| < \infty, \quad A \in \mathcal{L}_1(H).$$

But then $\sum \lambda_i = 0$, so that

$$\boxed{\lambda_1 + \lambda_2 + \cdots + \lambda_p = 0}. \tag{1}$$

Remark. – If $p = 2$, we get that $\lambda_2 = -\lambda_1$, i.e. $\sigma(G) = \{0, \pm r\} (r = r(G))$.

If $p = 3$, graph G has exactely one positive eigenvalue, and two negative eigenvalue $(\lambda_2, \lambda_3 < 0)$, so that $\lambda_1 = |\lambda_2| + |\lambda_3|$. □

If spectrum $\sigma(G)$ is finite, it is an important question too – how determine it.

We assume that for instance, columns $C_{i_1}, \dots, C_{i_p} (i_1 < i_2 < \cdots < i_p)$ are linearly independent, and all other depend on them.

Then non-zero spectrum $\sigma(G) \backslash 0$ coincides to the spectrum $\sigma(A \mid_{R(A)})$ of operator A in finite dimensional subspace $R(A)$.

Theorem 7. – *Non-zero eigenvalues $\lambda_1, \dots, \lambda_p$ of A are just roots of a "characteristic eguation"*

$$\det(B - \lambda I) = 0,$$

where $B = [b_{1m}]$ is a square $p \times p$ matrix, and

$$\begin{cases} b_{im} = \displaystyle\sum_{i=1}^{\infty} \quad c_{li} a_{ir_m} \\ Ae_i = \displaystyle\sum_{i=1}^{\infty} \quad c_{li} Ae_{rl} \quad (i = 1, 2, \dots) \end{cases}$$

Matrix B is regular and in general case non symmetric. □

We omit the proof.

EXAMPLE. – Let G be complete bipartite graph $K_{\{i\},\infty}$. Then its adjacency matrix is

$$A = \left[\begin{array}{cc|ccc} 0 & a & a^2 & a^3 & \dots \\ a & 0 & 0 & 0 & \dots \\ \hline a^2 & 0 & & & \\ a^3 & 0 & & 0 & \\ \dots & & & & \end{array}\right]$$

so that columns C_1, C_2 obviously form a basis in $R(A)$.

We obtain $c_{11} = c_{22} = 1, c_{12} = c_{21} = 0, C_{1i} = 0 (i \geq 3), c_{21} = a^{i-2} (i \geq 3), b_{11} = b_{22} = 0, b_{12} = a, b_{21} = a/(1 - a^2)$, and

$$B = \begin{bmatrix} 0 & a \\ \dfrac{a}{1 - a^2} & 0 \end{bmatrix}.$$

Then the eigenvalues are $\lambda = \pm a\sqrt{1 - a^2}$. □

8. Graphs with two non-zero eigenvalues.

We solve here a tipical problem of reconstruction of (infinite) graphs. Determine all graphs with a "small" number of non-zero eigenvalues (in case when this number is two).

If graph G has two non-zero eigenvalues λ_1, λ_2, then as we have seen,

$$\sigma(G) = \{0, \pm r\}, \ \ \textit{where} \ \ r = r(G).$$

THEOREM 8. *–All graphs with two non-zero eigenvalues $\pm r$ are the complete bipartite graphs $K(N_1, N_2)$.*

PROOF. – For simplicity, take that columns C_1, C_2 of matrix A are independent, and all other are their linear combinations (the proof is quite similar in a general case too).

We observe that $a_{12} \neq 0$, because in the opposite case, we can conclude that $a_{1i} = a_{2i} = 0 (i \in N)$, thus $C_1 = C_2 = 0$ which is impossible.

If now $C_i = pC_1 + qC_2 (i \geq 3, p = p_i, q_i)$ we have that

$$a_{1i} = qa, \ \ a_{2i} = pa \quad (p^2 + q^2 \neq 0),$$

so that $a_{i1} = qa$, $a_{i2} = pa$, but then

$$a_{ii} = pa_{i1} + qa_{i2} = 2pqa = 0,$$

so that $p = 0$ or $q = 0$.

Hence, every column of the adjacency matrix A is proportinal with C_1 or with C_2. If now $N_1 = \{i \mid C_i \sim C_1\}$ (including 1 too), and $N_2 = \{i \mid C_i \sim C_2\}$, by reconstruction the matrix A, one can can easily conclude that in the basis $\{e_i \mid i \in N_1\} \cup \{e_j \mid j \in N_2\}$ matrix A take the form

$$A = \begin{pmatrix} 0 & P' \\ P & 0 \end{pmatrix}$$

where P is a $N_2 \times N_1$ matrix whose all entries are distinct from zero.

Hence $G = K(N_1, N_2)$, q.e.d. □

– Finally, we only remark that spectral radii $r = r(G)$ of all complete bipartite graphs $K(N_1, N_2)$ cover the open segment $I = 0, d)$, where $d = 1/2(1 - a^2)$, so that their spectra cover the open segment $(-d, +d)$. □

We conclude with three open questions which we have concerned throughout the text.

1. – Does there exist two (non-isomorphic) isospectral graphs.

2. – Are the non-zero eigenvalues of every infinite graph G simple.

3. – Is there any non-bipartite graph whose spectrum is zero-symetric.

4. – Are all operators $A = A(G)$ corresponding to infinite graphs – nuclear, and then

$$\sum_{n=1}^{\infty} \lambda_n = 0.$$

Remark. As C. Godsil informed the author, recently B.D. Mc Kay in paper "The expected eigenvalue distribution of a random labelled regular graph" (Math. Reas. Reports Univ. Melbourne, rep. no–9 (1979), investigated the limits of eigenvalues of a class of finite regular graphs.

Acknowledgement. The author very thanks D.M. Cvetković for his interest and several valuable comments concerning this paper.

REFERENCES

[1] Ahiezer N.I. Glazman I. M., *Theory of linear operators in Hilbert space,* Višča škola Škola, Harkow, 1977.

[2] Collatz L., Sinogowitz W., *Spektren endlicher grafen*, Abh. Math. Sem. Univ. Hamburg – 21 (1957), 63-77.

[3] Cvetković D. M., *Graphs and their spectra* (Thesis), Univ. Belgrade Publ. El. tehn. fak. Ser. Math. Fys., No 354–356, (1971), 1–50.

[4] Cvetković D. M., Milić., *Teorija grafova i njene primene*, Naučna knjiga, Beograd, 1977.

[5] Cvetković D. M., Doob M., Sachs H., *Spectra of graphs-Theory and application*, Veb. Deutcher Verlag der Wiss., Berlin, 1980; Academic Press, New York, 1980.

[6] Dunford N., Schwartz. J. T., *Linear operators*, parts I, II, New York, 1963.

[7] Gohberg I. C., Krein M.G., *Introduction to the theory of linear non-selfadjoint operators*, Nauka, Moskva, 1965.

[8] Krein M.G. Rutman M.A., *Linear operators in a Banach space leaving a coonnas invariant*, Usp. Mat. Nauk – 3/1 (23), (1948), 3–95.

[9] Stecenko V. Ja., *On a spectral property of on irreducible operator*, UMN – 22/3 (135), (1967), 242–244.

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