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A NOTE ON SETS OF CONSTANT WIDTH

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Abstract. We prove here that each convex set in Euclidean space can be extended to a set of constant width having the same diameter and being contained in same Jung's ball. We also prove a characterization of the sets of constant width, which gives the answer to a problem of F. A. Valentine.

Introduction. In this note, we prove that each convex set C in Euclidean space can be extended to a set of constant width having the same diameter and being contained in the Jung's ball of the set C. An inequality between the radii of the corresponding Jung's ball and the inscribed ball and the diameter of the set C is proved.

In the third part of this note, we prove a characterization of the sets of constant width whose specialization gives the answer to a problem of F. A. Valentine [3., Problem 12.6.].

Our terminology and notations are according to F. A. Valentine [3]. Note that Jung's ball of the convex set C is the ball of the smallest diameter containing C. A set of constant width in a Minkowski space is defined in the usual manner, as a compact convex set for which every two parallel support hyper-planes are at the same distance apart. A complete set in a Minkowski space is a subset of the space whose each superset is of bigger diameter. In case of Euclidean spaces the class of complete sets and the class of sets of constant width coincide.

2. THEOREM 1. *Let C be a set of diameter d in Euclidean space E_n and $K(x,R)$ its Jung's ball. Then, there is a set B of constant width d such that*

$$C \subset B \subset K(x,R).$$

The author recently proved that it could be required for the set B (if $\operatorname{diam} C < 2R$) to satisfy the equality $B \cap S(x,R) = \bar{C} \cap S(x,R)$ too ($S(x,R)$ being the sphere of radius R about the center x). The proof will be published in the following paper.

PROOF. Consider the family

$$F = \{A \mid C \subset A \wedge A \subset K(x,R) \wedge \operatorname{diam} A = d\}$$

and the inclusion relation as an ordering on F. The family F is non-empty, because C is an element of F. Every chain in F has an upper bound, because the union of the elements of each chain in F is again an element of F. So by Zorn's lemma the family F has at least one maximal element. Let B be a manimal element of F. We shall prove that B is a set of constant width. Suppose the contrary and consider a set B_0 of constant width d, which contains B. The set B_0 is not contained in the ball $K(x,R)$ because of maximality of the set B in the family F

Let z be one of the farthest points of B_0 from the point x and let y be a point on the line through the points x and z chosen so that x is between z and y and that $\|x-y\| = d-R$. Because of $\|y-z\| = \|y-x\| + \|x-z\| = d-R+\|z-x\| > d$, it follows that $y \notin B_0$. Then, $y \notin B_0$ implies $y \notin B$. Because of $d \leq 2R$ we have $\|x-y\| = d-R \leq R$ and $y \in K(x,R)$. For $u \in B$ we have $u \in B \subset K(x,R)$ what implies $\|y-u\| \leq \|y-x\| + \|x-u\| \leq d-R+R = d$.

So, we have for the set $B' = \operatorname{conv}(B \cup \{y\})$

$$C \subset B \subset B' \text{ and } B' \subset K(x,R) \text{ and } \operatorname{diam} B' = d.$$

Hence, $B' \in F$ what contradicts the maximality of B. Hence, B is a set of constant width, which was to be proved.

Let us note that the same arguments would show the validity of Theorem 1. in case of a Minkowski space if the term "set of constant width" is replaced by the term "complete set", i.e. it holds

PROPOSITION 1. Let C be a set of diameter d in Minkowski space L_n and $K(x,R)$ one of the balls of the smallest diameter containing C. Then, there is a complete set B of diameter d such that

$$C \subset B \subset K(x,R).$$

Applying the Theorem 1, we give a simple proof of the following statement.

COROLLARY 1. Let C be a subset of E_n of diameter d with the radii of its Jung's ball and the inscribed ball being R and r respectively. Then, the inequality

$$r \leq d - R$$

holds.

PROOF. Let B be a set of constant width d such that

$$C \subset B \subset K(x,R),$$

where $K(x, R)$ is the Jung's ball of the set C, and let r' be the radius of the inscribed ball of the set B. Then, by Theorem 53. in [2], the inequality

$$r \leq r' = d - R$$

holds, which was to be proved.

3. THEOREM 2. *A compact convex body B in a Minkowski space L_n is the set of constant width if and only if for each pair of interior points $x, y \in int B$, there is a set C of constant width such that $x, y \in bd\, C$ and $C \subset int B$.*

PROOF. Let B be a set of constant width. Let $x, y \in \operatorname{int} B$. There is a set B' homothetic to B with y being the center of homothety such that $x \in bd\, B'$. There is now a set C homothetic to B' with x being the center of homothety such that $x, y \in bd\, C$ and $C \subset B' \subset \operatorname{int} B$.

Let B be a compact convex body which is not the set of constant width and $\operatorname{diam} B = d$. Then, there is a hyperplane H such that $w(H) = d' < d(w(H)$ being the distance between the two parallel hyperplanes of support to B which are parallel to $H)$. Because of $\operatorname{diam} B = d$, there exist points $x, y \in \operatorname{int} B$ such that $\|x - y\| > d'$. Then, every set of constant width which contains the points x and y has the width bigger then d' and it can not be contained in the set B.

REMARK. From the first part of the proof of Theorem 2. it is easy to see that the set C of constant width is homothetic to B. Hence, we shall have following

PROPOSITION 2. Let B' be any set of constant width in Euclidean space E_n.A compact convex body B in E_n is a set similar to B' if and only if for each pair of points x and y, being both either interior or exterior points of B, there exists a set C similar to B' such that x and y belong to $bd\, C$ and the sets C and B have disjoint boundaries.

PROOF. Let B be a set of constant width similar to B' and let x and y be the exterior points of B.

If the sets B and $[x, y]$ are disjoint, there is a hyperplane H which contains the point x and strictly separates the set B and the point y or contains the point y and strictly separates the set B and the point x. Let, for example, H contains x and strictly separates B and y. Then, there exists a set B, congruent to B such that $x \in bd\, B_1$ and H is a hyperplane of support to B_1 and y belongs to the interior of the cone of support to B_1 at the point x. So, there is a set C homothetic to B_1 with x being the center of homothety, and thus similar to B', such that $x, y \in bd\, C$ and $C \cap B = \emptyset$.

If the sets B and $[x, y]$ are not disjoint, then there is a set B_1 homothetic to B with an interior point of B being the center of homothety such that $x \in bd\, B_1$ and $y \notin \operatorname{int} B_1$ or $y \in bd\, B_1$, and $x \notin \operatorname{int} B_1$. Let be $x \in bd\, B_1$ and $y \notin \operatorname{int} B_1$. Because of $B \cap [x, y] \neq \emptyset$ the point y belongs to the interior of the cone of support to B_1 at the point x and so there is now a set C homothetic to B_1 with x being the center of homothety such that $x, y \in bd\, C$ and $B \subset \operatorname{int} C$ holds.

According to Theorem 2., that is all what we had to prove.

When the set B'is a ball, Proposition 2. gives one possible generalization of Proposition 12.14 in [3.], while Theorem 2. is a further generalization of this proposition, and is a solution of Problem 12.6 from [3].

REFERENCES

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