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PROPERTIES OF MONOTONE MAPPINGS IN PARTIALLY ORDERED SETS

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0. Introduction. The importance of monotone (increasing and decreasing) mappings in real analysis is well known. It appears that in studying many aspects of the theory of partially ordered sets the impotrance of monotone mappings is not less than in real analysis. In this paper we study some of these aspects with the stress on the existence of fixed points and the structure of the sets of fixed points.

First of all shall fix the terminology and notation. Let P be a partially ordered set. A self-mapping of P is a function $f : P \to P$. A point a is a fixed point of a self-mapping f of P if $f(a) = a$. The set of fixed points of a self-mapping f of P is denoted by $I(f, P)$. A mapping $f : P \to P'$, where P and P' are partially odreder sets, is *isotone* (or *increasing*) or *order-preserving* if, for any $x, y \in P \ \leq y$ implies $f(x) \leq f(y)$. The mapping f is *antitone* (or *decreasing* or *order-reversing* if, for any $x, y \in P$, $x \leq y$ implies $f(x) \geq f(y)$. If a p.o. set is such that every isotone self-mapping of it has at least one fixed point, then such a set is said to have the *fixed point property*. It is proved by A. Tarski and A. Davis that for a lattice the fixed point property is equivalent to lattice completeness. Many classes of p. o. sets have the fixed point property, but only trivial sets have property that any antitone self-mapping of them has a fixed point. Even two element chain (a totally ordered set) fail to have this property, since the mapping $f(a) = b, f(b) = a$ is a antitone without fixed point, for the chain $a < b$. From this reason we shall those antitone mapping which have some additional properties.

A p.o. set P is said to be of *finite length* provided that every chain C of P has a finite number of elements.

If $x, y \leq\in P$ and neither $x \leq y$ nor $y \leq x$, then we write $x \| y$.

The cardinality of a set A is denoted by $|A|$.

1. Some wanted properties. Let be a non-void set and $f_1, f_2, \dots, f_n$ self-mapping of P. We say that f is factored into $f_1, f_2, \dots, f_n$ provided that $f = f_1 \circ f_2 \circ \cdots f_n$. It is seen at once that a composition of increasing self-mappings of an ordered set P is also an increasing self-mapping of P, while a composition of decreasing self-mappings of P is increasing according to wether the number of mappings is even or odd.

If $g = f \circ f$, then f is said to be a square root of g. For example, the mapping defined on the chain $x_1 < x_2 < \cdots < x_n$ by $f(x_i) = x_{n+1-i}$ is a square root of the identity mapping on the chain.

Question 1. Let P be a partially ordered set and $f : P \to P$ an increasing mapping. Under which conditions there exists two decreasing self-mapings, f_1 and f_2, of P such that $f = f_1 \circ f_2$? In the special case under which conditions f is a square of a decreasing mapping g (i.e. $f = g \circ g$ for decreasing g)?

It is an easy matter to show that, if the set of fixed points of an antitone self-mapping of a partially ordered set is non-empty and has more than one point, then any two distinct points of this set are incoparable. Hence the set of fixed points of an antitone self-mapping of a partially ordered set P, if non-empty, forms an antichain.

Question 2. Let P be a partially ordered set $A \subset P$ an antichain. Under which conditions on P there exists an antitone $f : P \to P$ such that A is just the set of fixed points of f?

2. Antitone self-mappings of finite sets. We find some conditions under which an antitone self-apping of a finite set P has a unique fixed point.

Definition 2.1. Let $P \neq 0$, $f : P \to P$. The set P is said to be *$f-$producing* iff there exists a point $a \in P$ such that $P = \{a, f(a), f^2(a), \dots\}$. A set P is *partly f-producing* iff there is a non-void subset P' of P such that P' is f-producing.

THEOREM 2.1. *Let P be a finite chain. An antitone self-mapping of P has a unique fixed point if and only if P is partly f-producing and f does not permute two distinct points.*

Proof. We need the following lemma.

LEMMA 2.2. *Let P be a finite nonvoid chain, let $f : P \to P$ be antitone, let P be f-producing and f does not permute two distinct points. Then f has a unique fixed point.*

Proof of the lemma 2.2. If $|P| = 1$, there is nothing to prove. So take $|P| > 1$. Let $a \in P$. If $f(a) = a$, the proof is complete, otherwise $a < f(a)$ or $a > f(a)$. Suppose $a < f(a)$ (the case $f > f(a)$ is treated similarly). Then $f(a) \geq f^2(a)$. Equality $f(a) = f^2(a)$ proves the lemma. Let us take $f(a) > f^2(a)$. We distinguish two cases.

Case 1°. $f^2(a) < a$. It follows $f^3(a) > f(a)$ (equality $f^3(a) = f(a)$ is excluded, since f has no pair of permuted points.). Also $f^4(a) < f^2(a)$ and so on.

In this way we obtain the sequence

$$\cdots f^{2k}(a) < f^{2k-2}(a) < \cdots < f^2(a) < a < f(a) < f^3(a) < \cdots < f^{2k-1}(a) < \cdots$$

P being finite there exists an integer k such that $f^{2k}(a) = f^{2k+2}(a) = \cdots$ and $f^{2k-1}(a) = \cdots$ or $f^{2k+1}(a) = f^{2k+3}(a) = \cdots$).

But then points $f^{2k}(a)$ and $f^{2k-1}(a)$ (or $f^{2k}(a)$ and $f^{2k+1}(a)$) are permuted. So this case cannot occure.

Case 2°. $a \leq f^2(a)$. Then $f(a) \geq f^3(a) \geq f^2(a) \geq a$. Applying repeatedly f we obtain

$$a \leq f^2(a) \leq f^4(a) \leq \cdots \leq f^{2k}(a) \leq \cdots \geq f^{2k+1}(a) \leq f^{2k-1}(a) \leq f^3(a) \leq f(a).$$

Since P is finite, there exists an integer m such that $f^{2m}(a) = f^{2m+2}(a) = \cdots$ and $f^{2m-1}(a) = f^{2m+1}(a) = \cdots$ (or $f^{2m+1}(a) = f^{2m+3}(a) = \cdots$).

Supposition $f^{2m}(a) \neq f^{2m-1}(a)$ (or $f^{2m}(a) \neq f^{2m+1}(a)$) contradict the fact that f is non-permuting. Hence, there exists an integer m such that $f^{2m}(a) = f^{2m-1}(a)$ (or $f^{2m}(a) = f^{2m+1}(a)$). It follows that $f^{m-1}(a)$ (or $f^{2m}(a) = f^{2m+1}(a)$) is a fixed point.

Let us note first that the fixed point (if it exists) is unique. If we suppose that there are two, x and y say, we would, for example, have $x < y$, hence $x = f(x) \geq f(y) = y$, a contradiction.

Proof of the theorem 2.1. Since $f(A) \subset A$, where A is finite chain, the sufficiency of the condition follows from the above lemma.

Necessity is obvious namely, if a is a fixed point of f, then $f(a) = a$ nad P is partly f-producing for $P' = \{a\}$.

THEOREM 2.3. *Let P be a partially ordered set of finite lenght and f an antitone self-mapping of P such that $f(P) = C$, where C is a maximal chain of P nad C is partly f-producing. Then f has a unique fixed point.*

Proof. If such antitone self-mapping f of P exists, then the existence and unicity of the fixed point of f follows from the previous theorem, where P from the previous theorem is replaced by C. The only thing to be proved is that the set of self-mappings is non-empty. We shall construct one such mapping.

Let $C = \{x_1, \dots, x_n\}$ be a maximal chain of P, where $x_1 < x_2 < \cdots < x_n$. Let us set $f(x_i) = x_{n+1-i}$. To define f on the rest of P we shall define an isotone self-mapping g of P and take f to be a square root of g. For $x \in C$ we set $g(x) = x$.

Let $x \in P$. Set

$$C\{c \in C | x \| c\}$$

Evidently $C_x = \varnothing$ if $x \in C$. Conversely if $C_x = \varnothing$, then $C \cap \{x\}$ is a chain, so $x \in C$, by maximality of C.

Note also, that for $a, b \in P \backslash C$,

(*) $$a < b \text{ implies } \min C_a \leq \min C_b$$

In fact supposing min $C_b <$ min C_a, we would have

$$\max\{c \in C | c \leq b\} < \max\{c \in C | c \leq a\}$$

which is impossible.

For any $x \in P \backslash C$ set $g(x) = \min\ C_x$. Let us verify that g is isotone.

If $a < b$ and $a, b \in P \backslash C$, then from $(*)$ it follows that $g(a) \leq g(b)$.

Suppose now that only a (or b) is in C.

If $a \in C$, $b \in P \backslash C$, then $a < c$ for any $c \in C_b$, hence $a = g(a) < g(b)$. If $a \in P \backslash C$, $b \in C$, then $c < b$ for any $c \in C_a$, hence again $g(a) = \min\ C_a < b = (g)b$.

In this way, for any $a, b \in P$, $a \leq b$ implies $g(a) \leq g(b)$. We now define f to be a square root of g. Evidently f is antitone and sends P into C.

3. Some antitone self-mappings with unique fixed points. In our paper [**2**] we proved the following theorem

THEOREM 3.1. *Let P be a complete lattice and $g, h : P \to P$ two isotone conjugately factorable isotone mappings. Then*

(i) $f_1(I,\ h,\ P)) \subset I(g,\ P)$

(ii) $f_2(I(g,\ P)) \subset I(h,\ P)$.

Taking $f_1 = f_2 = f$ we can state the following

THEOREM 3.2. *Let P be a complete lattice and $f : P \to P$ an antitone mapping. Then*

$$f(I(f^2,\ P) \subset I(f^2,\ P)$$

Using this theorem we shall prove the following

THEOREM 3.3. *P be a complete lattice and $f : P \to P$ an antitone mapping such that one of the following conditions is fulfilled:*

(C) *For any $x \in I(f^2,\ P)$ is $x \geq f(x)$ or $x \| f(x)$*

(C') *for any $x \in I(f^2,\ P)$ is $x \leq f(x)$ or $x \| f(x)$*

Then f has a unique fixed point.

Proof. According to Tarski's theorem the set $I(f^2,\ P)$ is non-empty and contains a least element, m say. By theorem 3.2 $f(m) \geq m$, and by condition (C) the sign $>$ is excluded, so $f(m) = m$.

To prove the uniqureness of the fixed point of f, we assume that c is another fixed point of f. Then $c \in I(f^2,\ P)$, hence $c \geq m$, which implies $c = f(c) \leq f(m) = m$, or $c = m$.

The existence and uniqueness of fixed point of f satisfying (C') is proved using the greatest element, n say of $I(f^2, P)$.

Remark. The relevance of antitone mappings satisfying Condition (C) is pointed out in [**4**].

Let us now examine the structure of some subsets of $I(f^2,P)$. Provided that P is a complete lattice and f an antitone self-maping of P.

Put

$$S=\{x\in I(f^2,\ P)|x\leq f(x)\}$$
$$S'=\{x\in I(f^2,\ P)|x\geq f(x)\}.$$

One easily proves the following

PROPOSITION 3.4. *If $x\in S$ and $y<x$, then $y\in S$. Also if $x\in S'$ and $y>x$, then $y\in S'$.*

We have the following

PROPOSITION 3.5. *If P is a complete lattice and f an antitone self-mapping of P, then the set S is left complete semi-lattice, while the set S' is a right complete semi-lattice(subsemi-lattice of $I(f^2,\ P)$).*

Proof. By theorem 3.3. both sets are non-empty. Let $A\subset S,\ A\neq\varnothing$ and $a=\inf\ A$ (which exists in $I(f^2,\ P)$ according to Tarski's theorem). Then for any $x\in A$ is $a\leq x$. Since f is antitone and $x\in S$, it follows $f(a)\geq f(x)\geq x\geq a$, hence $a\in S$, proving the first part of the proposition. The second part is proved analogously.

4. Connection between $I(f,\ P)$ and $I(f^2,\ P)$.

If P is a partially ordered set and $f:P\to P$ antitone, then $f^2:P\to P$, being isotone, has much more possibilities to have fixed points. We now study the connection between fixed points of f^2 and fixed points of f (if they exist).

A very easy and special case is when $I(f^2,\ P)$ contains only one point. Then this point is also fixed point of f.

In the following we asume f antitone and formulate a necessary and sufficient condition for an element of $I(f^2,\ P)$ to be an element of $I(f,\ P)$.

For any $a\in P$ let us consider the set

(1) $$S_a=\{f^\circ(a)=a,\ f(a),\ f^2(a),\dots\}$$

and itroduce in S_a the operation o defined by

$$f^k(a)\circ f^m(a)=f^{k+m}(a)$$

Denote by (f,a) the so obtained semigroup and by $\mathcal{M}$ the set all (f,a), for $a\in P$, which are not of order 2. Also we shall write $M(f,\ P)$ the union of all elements of $\mathcal{M}$.

THEOREM 4.1. *Let P be a partially ordered set, $f:P\to P$ an antitone mapping and $I(f^2,\ P)\neq\varnothing$. Then $I(f,\ P)\neq\varnothing$ if and only if $\mathcal{M}\neq\varnothing$ and $I(f^2,\ P)\cap M(f,\ P)\neq\varnothing$. Moreover $I(f,\ P)=I(f^2,\ P)\cap M(f,\ P)$.*

Proof. Suppose $I(f^2,\ P) \neq \varnothing,\ \mathcal{M} \neq \varnothing$ and $I(f^2,\ P) \cap M(f,\ P) \neq \varnothing$. Let a be an element of the last set. Then $a = f^2(a)$ and the set (1) is reduced to $\{a, f(a)\}$. But $(f, a) \in \mathcal{M}$, hence $|\{a,\ f(a)\}| = 1$, or $a = f(a)$, that is $a \in I(f,\ P)$.

Conversely, let $a = f(a)$. Then the set (1) is reduced to $\{a\}$. The set $M(f,\ P)$ is non-empty and contains at least one element of $I(f^2, P)$. This is true for every element in $I(f,\ P)$. Every element in $I(f^2,\ P) \cap M(f,\ P)$ and no element of the last set is out of $I(f, P)$.

We now take P to be a complete lattice and $f : P \to P$ an antitone maping. Then

(i) The set $I(f^2, P)$ is non-empty;

(ii) $f(I(f^2,\ P)) \subset I(f^2,\ P)$.

The assertion (i) is one of the conclusions of Tarski's theorem, since f^2 is isotone, and (ii) is a consequence of our theorem 3.2 (see [**2**]).

Let us give the direct proof of (ii). If $x \in I(f^2,\ P)$, then $f^2(f(x)) = f(f^2(x) = f(x)$, hence $f(x) \in I(f^2,\ P)$, as required.

This proof of (ii) makes no use of the fact that P is a complete lattice. The only fact of importance is that $I(f^2,\ P) \neq \varnothing$. So we have the following theorem.

THEOREM 4.2. *Let P be a non-void partially ordered set, $f : P \to P$ antitone and let $I(f^2,\ P) \neq \varnothing$. Then $f(I(f^2,\ P)) \subset I(f^2,\ P)$.*

We now remind the definition of a join antimorphism. Let P be a complete lattice and $f : P \to P$ such that, for any $A \subset P,\ A \neq \varnothing$,

$$f(\sup\ A) = \inf\ f(A)$$

Such a self-mapping of P is refered to as a *join antimorphism*.

Taking $A = \{a,\ b\}$, with $a \leq b$, we find that, if f is a join antimorphism, $a \leq b$ implies $f(a) \geq f(b)$, i. e. a *join antimorphism is an antitone mapping*. On the other hand, not every antitone mapping is a join antimorphism.

By Tarski's theorem [**1**], under the suppositions on the theorem 3.2, $m = \min\ I(f^2,\ P))$ exists and from the theorem 3.2 we infer $m \leq f(m)$, so the following corollary is valid.

COROLLARY 4.3. *(A. E. Roth [**4**]). Let P be a complete lattice and $f : P \to P$ a join antimorphism. Then there exists $x \in P$ such that $x = f^2(x)$ and $x \leq f(x)$.*

In the following comentary we shall improve the above Roth's theorem. First of all f may be any antitone mapping, not necessarily a join antimorphism. Besides, if $|I(f^2,\ P)| > 1$, with an antitone f, $x < f(x)$ *for at least one* $x \in I(f^2, P)$. In fact, if $m = \min\ I(f^2,\ P)$ and $m = f(m)$, then for any $x > m$ (such an x exists, according to the supposition on $I(f,\ P)$ it would be $f(x) \leq m$. Thorem 3.2 yields $f(x) = m$ and $f^2(x) = x = f(m) = m$; a contradiction.

Denote by m and n the least and the greatest element of the lattice $I(f^2,\ P)$.

Proposition 4.4. *Let P be a complete lattice, $f : P \to P$ an antitone mapping. If one of the conditions is satisfied:*

(a) $m \in I(f,\ P)$, *or*

(b) $n \in I(f,\ P)$,

then $|I(f^2,\ P)| = 1$.

Proof. (a) Suppose $|I(f^2,\ P)| > 1$. Then there exists $x \in I(f^2,\ P)$, such that $x > m$. It follows $f(x) \leq f(m)$, or $f(x) \leq m$. From the Theorem 3.2 we find $f(x) \geq m$, hence $f(x) = m$. Since $x \in I(f^2,\ P)$ we have $x = f^2(x) = f(m) = m$, which is a contradiction.

The proof of (b) is analogous.

Let P be a partially ordered set and $f : P \to P$ be a bijection and antitone. Then f is said to be a *d-automorphism of* P.

Not every partially ordered set admits a d-automorphism.

Example. Let $P = \{a, b, c\},\ a < b,\ a < c,\ b \| c$. Then no d-automorphism of P exists.

A partially ordered set having at least one d-automorphism is said to be a *s-set*. In [**3**] more examples of sets are constructed.

It is clear that the set of fixed points of an isotone self-mapping of a partially ordered set need not be a s-set. In contrast to this fact we have the following theorem.

Theorem 4.5. *Let P be a partially ordered set having fixed point property. Then for any antitone $f : P \to P$ the set $I(f^2,\ P)$ is a s-set,*

Proof. By the supposition on P, $I(f^2,\ P) \neq \varnothing$ and by Theorem 3.2 $f|I(f^2,\ P)$ is a self-mapping of $I(f^2,\ P)$.

We need the following notion. Let P be a partially ordered set and $f : P \to P$. A point a of P is said to be *permuted by* f iff there exists another point b of P such that $f(a) = b,\ f(b) = a$.

To finish the proof we shall show that every point of $I(f^2,\ P)$ is either a fixed point of f or a permuted point of f. Let $x \in I(f^2,\ P)$. Put $f(x) = y$. Then $f(y) = f((f(x)) = x$, hence x is a permuted point by f or $y = x$.

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