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MINIMAL MODAL SYSTEMS IN WHICH HEYTING AND CLASSICAL LOGIC CAN BE EMBEDDED

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By a *translation* from a system S_1 into a system S_2 we shall understand a mapping from formulae of S_1 into formulae of S_2; and we shall say that S_1 can be *embedded* in S_2 by a translation t (from S_1 into S_2) iff $[\vdash_{S_1} A$ iff $\vdash_{S_2} t(A)]$.

It is well known (v. e.g. Czermak 1975) that the Heyting and classical propositional calculi can be embedded in the propositional calculi $S\,4$ and $S\,5$ respectively, by the following translation (we shall use "A, B ..., $A_1, \dots$" as schemata for propositional formulae)

$$\begin{aligned}
t_1(A) &= \Box A, \quad \text{where } A \text{ is an atomic formula } (\bot \text{ and } \top \text{ are also atomic formulae})\\
t_1(A \to B) &= \Box(t_1(A) \to t_1(B))\\
t_1(A \wedge B) &= \Box(t_1(A) \wedge t_1(B))\\
t_1(A \vee B) &= \Box(t_1(A) \vee t_1(B))\\
t_1(\rceil A) &= \Box \rceil t_1(A);
\end{aligned}$$

i.e. t_1 prefixes $\Box$ to every subformula.

This is a variant of the McKinsey-Tarski translation. Another variant, derived from McKinsey & Tarski 1948, is obtained from t_1 by substituting "t_2" for "t_1" everywhere except in the clauses for $\wedge$ and $\vee$, where we have

$$\begin{aligned}
t_2(A \wedge B) &= t_2(A) \wedge t_2(B)\\
t_2(A \vee B) &= t_2(A) \vee t_2(B).
\end{aligned}$$

These two variants are equivalent for $S\,4$ and $S\,5$ since in both of these systems we have as theorems

(a) $$\Box(\Box A \wedge \Box B) \leftrightarrow (\Box A \wedge \Box B)$$

(b) $$\Box(\Box A \vee \Box B) \leftrightarrow (\Box A \vee \Box B).$$

Systems stronger than $S\,4$ in which Heyting logic can be embedded by t_1, or t_2, have also been studied. The system sometimes named "$S\,4$ *Grz*" (after Grzegorczyk 1967), which is important for the "probability interpretations of modal logic" (v. Boolos 1979), is the strongest in a family of such extensions of $S\,4$, studied by Esakia 1974 and 1979.

In this paper we shall invetigate systems weaker than $S\,4$ and $S\,5$ in which Heyting classical logic, respectively, can be embedded by the translation t_1. We shall rely for this investigation on some already known results of modal logic. Our notation and terminology will try to follow that of Chellas 1980.

A modal will be called *normal* iff the set of its theorems is closed under

$$\text{MP.}\ \frac{A \quad A \to B}{B}$$

$$\text{RN.}\ \frac{A}{\Box A}$$

and substitution, and contains all tautologies and all the formulae

$$\text{K.}\quad \Box(A \to B) \to (\Box A \to \Box B).$$

Any normal system is closed under

$$\text{RK.}\ \frac{A_1 \to (A_2 \to \cdots \to (A_{n-1} \to A_n)\cdots)}{\Box A_1 \to (\Box A_2 \to \cdots \to (\Box A_{n-1} \to \Box A_n)\cdots)}$$

$$\text{REP.}\ \frac{B \leftrightarrow B'}{A \leftrightarrow A[B/B']},$$

where $A[B/B']$ results from A by replacing zero or more occurrences of B in A by B', and has all the formulae

$$\text{R.}\quad \Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$$

as theorems (v. Chellas 1980, Chapter 4).

The system KD 4!

Take the language of the propositional calculus with $\rightarrow$, $\perp$ and $\Box$ as primitive constants ($\wedge$, $\vee$, $\rceil$, $\top$ and $\leftrightarrow$ are defined as usual in classical logic; "$\Diamond A$" is defined as "$\Box(A \rightarrow\perp) \rightarrow\perp$"). KD 4! is axiomatized by adding to a sufficient axiomatic basis for the classical propositional calculus with MP. the following rules and axion-schemata

RN. $\dfrac{A}{\Box A}$

K. $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

D. $\Box A \rightarrow \Diamond A$

4. $\Box A \rightarrow \Box\Box A$

4 c. $\Box\Box A \rightarrow \Box A$

("$\Box\Box A \leftrightarrow \Box A$" is called "4!").

It is clear that KD4! is normal. For any normal system we have that D. can be replaced by

$$D \perp . \quad \Box \perp\rightarrow\perp,$$

for we have

$$\begin{aligned}(\Box A \rightarrow (\Box(A \rightarrow\perp) \rightarrow\perp)) &\leftrightarrow ((\Box A \wedge \Box(A \rightarrow\perp)) \rightarrow\perp)\\ &\leftrightarrow (\Box A \wedge (A \rightarrow\perp)) \rightarrow\perp)\\ &\leftrightarrow (\Box \perp\rightarrow\perp)\end{aligned}$$

Standard models for KD4!

Let a *standard model* of modal propositional logic $\mathcal{M} = \langle W, R, P\rangle$ be defined as in Chellas 1980 (pp. 67 ff): W is a set of "worlds", R is a binary relation on W (i.e. $R \subseteq W \times W$), and P is a function from atomic formulae, without $\perp$, into $\mathcal{P}W$. The conditions for A being true at an $\alpha \in W$ in $\mathcal{M}$ $\left(\text{i.e. for } \overset{\mathcal{M}}{\underset{\alpha}{\models}} A\right)$ are given as usual:

$$\overset{\mathcal{M}}{\underset{\alpha}{\models}} A \quad \text{iff} \quad \alpha \in P(A) \subseteq W, \text{ where } A \text{ is atomic and is not } \perp$$

$$\overset{\mathcal{M}}{\underset{\alpha}{\models}} A \rightarrow B \quad \text{iff} \quad \left[\text{if } \overset{\mathcal{M}}{\underset{\alpha}{\models}} A \text{then } \overset{\mathcal{M}}{\underset{\alpha}{\models}} A\right]$$

$$\text{not } \overset{\mathcal{M}}{\underset{\alpha}{\models}} \perp$$

$$\overset{\mathcal{M}}{\underset{\alpha}{\models}} \Box A \quad \text{iff} \quad \forall\beta \in W,\ \alpha R\beta \cdot \overset{\mathcal{M}}{\underset{\beta}{\models}} A.$$

A formula A is valid (i.e. $\models A$) iff for every $\mathcal{M}\forall\alpha \in W \cdot \overset{\mathcal{M}}{\underset{\alpha}{\models}} A$.

Relying on results of Lemmon & Scott 1977 (Section 4) it can be shown that

$$\vdash_{KD4!} A \quad \text{iff} \quad \models A$$

with respect to models $\mathcal{M}$ where R is

(1) *serial*, i.e. $\forall \alpha \in W \exists \beta \in W \cdot \alpha R \beta$

(2) *transitive*, i.e. $\forall \alpha, \beta, \gamma \in W$ [if $\alpha R \beta$ and $\beta R \gamma$, then $\alpha R \gamma$]

(3) (*weakly*) *dense*, i.e. $\forall \alpha, \gamma \in W$ [if $\alpha R \gamma$, then $\exists \beta \in W[\alpha R \beta$ and $\beta R \gamma]]$

((2) and (3) give together that $R^2 = R$).

Relying on this modelling it can easily be shown that not every instance of $\Box A \to A$ is a theorem of KD4!, and hence that KD4! is property contained in $S4$.

However, KD4! is clodsed under the rule

$$\text{RNc.} \frac{\Box A}{A}.$$

To show that, suppose that not $\vdash_{KD4!} A$. Hence, there is a model $\mathcal{M}$ and an $\alpha \in W$ such that not $\models^{\mathcal{M}}_{\alpha} A$. Now extend $\mathcal{M}$ to $\mathcal{M}'$ so that $W' = W \cup \{\beta\}$ and

$$\begin{aligned} \forall \gamma, \delta \in W'[\gamma R' \delta \text{ iff } &[\gamma R \delta \text{ or } [\gamma = \beta \text{ and } \delta = \beta] \\ &\text{or } [\gamma = \beta \text{ and } \delta = \alpha] \text{ or } [\gamma = \beta \text{ and } \alpha R \delta]]]. \end{aligned}$$

It is easy to show for $\mathcal{M}'$ that R' is serial, transitive and (weakly) dense. Since not $\models^{\mathcal{M}}_{\beta} A \Box A$, not $\vdash_{KD4!} \Box$ (cf. Chellas 1980, p. 99).

For the modalities of KD4!, which are closely related to the modalities of S4, consult Chellas 1980 (pp. 155, 170).

The system H

Let H be the Heyting propositional calculus based on $\to, \wedge, \vee, \bot, \top$, and $\rceil$, and axiomatized by

MP. $\dfrac{A \quad A \to B}{B}$

a1. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

a2. $A \to (B \to A)$

a3. $(C \to A) \to ((C \to B) \to (C \to (A \wedge B)))$

a4. $(A \wedge B) \to A$

a5. $(A \wedge B) \to B$

a6. $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$

a7. $A \to (A \vee B)$

a8. $B \to (A \vee B)$

a9. $\bot \to A$

a10. $\top$

a11. $(A \to B) \to (\rceil B \to \rceil A)$

a12. $A \to \rceil\rceil A$

a13. $\rceil\rceil A \to (\rceil A \to A)$

(v. Kanger 1955).

We can show

Lemma 1. *If* $\vdash_{H} A$, *then* $\vdash_{KD4!} t_1(A)$.

Demonstration: By induction on the length of proof of A in H. If A is al. we have

$$\begin{array}{l}
\Box(\Box C \to \Box D) \to (\Box\Box C \to \Box\Box D) \\ \hline
\Box(\Box C \to \Box D) \to (\Box C \to \Box D) \\ \hline
\Box B \to \Box(\Box C \to C \to \Box D)) \to (\Box B \to (\Box C \to \Box D)) \\ \hline
(\Box B \to \Box(\Box C \to \Box D)) \to ((\Box B \to \Box C) \to (\Box B \to \Box D)) \\ \hline
\Box(\Box B \to \Box(\Box C \to \Box D)) \to (\Box(\Box B \to \Box C) \to \Box(\Box B \to \Box D)) \\ \hline
\Box\Box(\Box B \to \Box(\Box C \to \Box D)) \to \Box(\Box(\Box B \to \Box C) \to \Box(\Box B \to \Box D)) \\ \hline
\Box(\Box(\Box B \to \Box(\Box C \to \Box D)) \to \Box(\Box(\Box B \to \Box C) \to \Box(\Box B \to \Box D))).
\end{array}$$

We proceed analogously for a2.—a13. Then we can use the theorems proved that way because $t_1(D)$ is always of the form $\Box E$ for some E.

If $\vdash_{KD4!} t_1(B)$ and $\vdash_{KD4!} t_1(B \to C)$, then we have

$$\dfrac{\dfrac{\dfrac{T_1(B)}{\Box t_1(B)} \qquad \dfrac{\Box(t_1(B) \to t_1(C))}{\Box t_1(B) \to \Box t_1(C))}}{\Box t_1(C)}}{t_1(C)}$$

since $t_1(C)$ is $\Box\, D$ for some D. Q.E.D

Negation is best treated separately in H, and not as defined with $\to$ and $\bot$, because $t_1(\urcorner A) = \Box\urcorner t_1(A) = \Box(t_1(A) \to \bot)$, whereas $t_1(A \to \bot) = \Box(t_1(A) \to \Box\bot)$. However, due to $\Box\bot \leftrightarrow \bot$, this last formula is equivalent with $\Box(t_1(A) \to \bot)$ in KD4!.

Since if $\vdash_{KD4!} t_1(A)$, then $\vdash_{S4} t_1(A)$, it follows from Lemma 1 and the embedding theorem for S4 that

Theorem 1. $\vdash_{H} A$ *iff* $\vdash_{KD4!} t_1(A)$.

Though

(a) $\quad \Box(\Box A \wedge \Box B) \leftrightarrow (\Box A \wedge \Box B)$

(b") $\quad (\Box A \vee \Box B) \to \Box(\Box A \vee \Box B)$

are theorems of KD4!,

(b') $\quad \Box(\Box A \vee \Box B) \to (\Box A \vee \Box B)$

is not. This can be shown with a suitable model invalidating (b'). This can serve to show that H *cannot* be embedded in KD4! by the translation t_2. For suppose it can; then since

$$\vdash_{H} (\top \to (A \vee B)) \to (A \vee B),$$

where A and B are atomic, we have

$$\vdash_{KD4!} \Box(\Box(\Box\top \to (\Box A \vee \Box B)) \to (\Box A \vee \Box B)).$$

And since in every normal system $\vdash \Box\top \leftrightarrow \top$ and $\vdash (\bot \to C) \leftrightarrow C$, we have

$$\vdash_{KD4!} \Box(\Box(\Box A \vee \Box B) \to (\Box A \vee \Box B)).$$

But KD4! is closed under RNc., as we have shown, and so we get a contradiction.

For (b'), and the corresponding condition on R for models of systems having (b'), consult Lemmon & Scott 1977 (pp. 68–71). Note that (b') entails 4 c. for normal systems.

The system KD45

Everything is as for KD4! except that instead of 4 c. we have the axiom-schema

5. $(\Box A \to \bot) \to \Box(\Box A \to \bot)$.

4 c. is a theorem of KD45, for we have

$$\dfrac{\dfrac{\dfrac{(\Box A \to \bot) \to \Box(\Box A \to \bot)}{(\Box A \to \bot) \to (\Box\Box A \to \Box\bot)}}{(\Box A \to \bot) \to (\Box\Box A \to \bot)}}{\Box\Box A \to \Box A.}$$

Standard models for KD45

It can be shown that

$$\vdash_{KD45} A \quad \text{iff} \quad \models A$$

with respect to models $\mathcal{M}$ where R is

(1) *serial*

(2) *transitive*

(3) *euclidean*, i.e.$\forall \alpha, \beta, \gamma \in W$ [if $\alpha R \beta$ and $\alpha R \gamma$, then $\beta R \gamma$]

(v. Chellas 1980, Chapter 5).

It can also be shown that not every instance of $\Box A \to A$ is a theorem of KD45, and hence that KD45 is properly contained in S5 (v. Chellas 1980, p. 168).

For the modalities of KD45, which are closely related to the modalities of S5, consult Chellas 1980 (pp. 154, 169).

Contrary to KD4!, KD45 is *not* closed under RNc. (v. Chellas 1980, p. 168). However, we can shown the following

LEMMA 2. 2.1. *If $\Box A$ is $t_1(B \to C)$, or $t_1(B \wedge C)$, or $t_1(\rceil B)$, for some B and C of the language of* H, *then* $\vdash_{KD4!} \Box A \to A$.

2.2. *If $\Box A$ is $t_1(B \vee C)$, for some $\vdash_H B \vee C$, then $\vdash_{KD4!} \Box A \to A$.*

2.3. *If $\Box A$ is $t_1(B \vee C)$, for some B and C of the language of* H, *then* $\vdash_{KD45} \Box A \to A$.

Demonstration: 2.1. We have

$$\frac{\Box(\Box A_1 \to \Box A_2) \to (\Box\Box A_1 \to \Box\Box A_2)}{\Box(\Box A_1 \to \Box A_2) \to (\Box A_1 \to \Box A_2);}$$

$$\frac{\Box(\Box A_1 \wedge \Box A_2) \to (\Box\Box A_1 \wedge \Box\Box A_2)}{\Box(\Box A_1 \wedge \Box A_2) \to (\Box A_1 \wedge \Box A_2);}$$

$$\frac{\Box(\Box A_1 \to \bot) \to (\Box\Box A_1 \to \Box \bot)}{\Box(\Box A_1 \to \bot) \to (\Box A_1 \to \bot);}.$$

2.2. If $\vdash_H B \vee C$, then either $\vdash_H B$ or $\vdash_H C$. If $\Box A$ is $\Box(\Box A_1 \vee \Box A_2)$, then by Lemma 1, either $\vdash_{KD4!} \Box A_1$ or $\vdash_{KD4!} \Box A_2$. Hence, $\vdash_{KD4!} \Box A_1 \vee \Box A_2$, and $\vdash_{KD4!} \Box(\Box A_1 \vee \Box A_2) \to (\Box A_1 \vee \Box A_2)$.

2.3. We have

$$\frac{\Box(\rceil\Box A \to \Box B) \to (\Box\rceil\Box A \to \Box\Box B)}{\Box(\rceil\Box A \to \Box B) \to (\rceil\Box A \to \Box B),}$$

i.e. (b') is a theorem of KD45. Q.E.D.

The system C

C will be the classical propositional calculus specified exactly like H, save that it has in addition the axiom-schema

$$\text{a}\,14.\ \ ((A \to B) \to A) \to A.$$

We can show

LEMMA 3. *If $\vdash_C A$, then $\vdash_{KD45} t_1(A)$.*

Demonstration. We proceed as in the demonstration of Lemma 1 with an additional case in the basis of the induction. If A is a14., we have

$$\cfrac{\cfrac{\cfrac{\rceil\Box A \to (\Box A \to \Box B)}{\Box\rceil\Box A \to \Box(\Box A \to \Box B)}}{\rceil\Box A \to \Box(\Box A \to \Box B)} \qquad \cfrac{\cfrac{\Box B \to (\Box A \to \Box B)}{\Box\Box B \to \Box(\Box A \to \Box B)}}{\Box B \to \Box(\Box A \to \Box B)}}{\cfrac{(\Box A \to \Box B) \to \Box(\Box A \to \Box B)}{\cfrac{(\Box(\Box A \to \Box B) \to \Box A) \to ((\Box A \to \Box B) \to \Box A)}{\cfrac{(\Box(\Box A \to \Box B) \to \Box A) \to \Box A}{\Box(\Box(\Box(\Box A \to \Box B) \to \Box A) \to \Box A).}}}}$$

Q.E.D.

Since if $\vdash_{KD45} t_1(A)$, then $\vdash_{S5} t_1(A)$, it follows from Lemma 3 and the embedding theorem for S5 that

THEOREM 2. $\vdash_{C} A$ *iff* $\vdash_{KD45} t_1(A)$.

Since we have as theorems in KD45 (a) and (b) (v. the demonstration of Lemma 2.3), t_2 could also be used to embed C in KD45. Furthermore, since

$$\vdash_{KD45} \Box(\Box A \rightarrow \Box B) \leftrightarrow (\Box A \rightarrow \Box B)$$
$$\vdash_{KD45} \Box\rceil\Box A \leftrightarrow \rceil\Box A$$

(v. the demonstrations of Lemmata 2.1 and 3), we could use for that embedding the translation t_3, which differs from t_2 in having

$$t_3(A \rightarrow B) = t_3(A) \rightarrow t_3(B)$$
$$t_3(\rceil A) \ \ = \rceil t_3(A);$$

i.e. in t_3 $\Box$ is prefixed only to atomic formulae, the translation of the rest being literal. Of course, this embedding is somewhat trivial since **C** can be embedded by t_3 (as well as by the literal translation of every formula) in *any* normal system.

* * *

The minimal modal systems in which the Heyting and classical propositional calculi can be embedded by t_1 are, respectively, those which have as theorems only $\{t_1(A) | \vdash_{H} A\}$ and $\{t_1(A) | \vdash_{C} A\}$. Let us call these two systems "SH" and "SC". SH can be axiomatized by taking as axioms all the formulae $t_1(A)$, where A is an axiom of H, and as a rule

$$\frac{t_1(A) \qquad\qquad t_1(A \rightarrow B)}{t_1(B)}.$$

We proceed analogously with SC. SH and SC do not have non-modal theorems, and are hence weaker than KD4! and KD45, respectively.

Let us say that a system is axiomatized in the *Lemmon style* if to a basis for the non-modal calculus are added modal axioms or rules. It is obvious that SH and SC cannot be axiomatized in the Lemmon style. A *fortiori*, SH and SC are not normal.

If we take into account axiomatization in the Lemmon style and normality, there is a sense in which KD4! and KD45 are minimal modal systems in which we can embed H nad C, respectively:

THEOREM 3 3.1 KD4! *is the minimal normal modal system closed under* $\frac{\Box A}{A}$, *where* $\Box A$ *is* $t_1(B)$ *for some* $\vdash_{H} B$, *in which* H *can be embedded by* t_1.

3.2 KD45 *is the minimal normal modal system closed under* $\frac{\Box A}{A}$, *where* $\Box A$ *is* $t_1(B)$ *for some* $\vdash_{C} B$, *in which* C *can be embedded by* t_1.

Demonstration: 3.1 Let S be a normal modal system in which H can be embedded by t_1. Since we have

$$\vdash_H \neg\bot,$$

we must have

$$\vdash_S \Box\neg\Box\bot, \qquad \text{i.e.}$$
$$\vdash_S \Box(\Box\bot\to\bot),$$

which by the closure condition for S gives

$$\vdash_S \Box\bot\to\bot\ .$$

Since we have

$$\vdash_H A\to(\top\to A)$$
$$\vdash_H (\top\to A)\to A,$$

where A is atomic, we must have

$$\vdash_S \Box(\Box A\to\Box(\Box\top\to\Box A))$$
$$\vdash_S \Box(\Box(\Box\top\to\Box A)\to\Box A),$$

which by the closure condition for S gives

$$\vdash_S \Box A\to\Box(\Box\top\to\Box A)$$
$$\vdash_S \Box(\Box\top\to\Box A)\to\Box A.$$

Since S is normal, we have $\vdash_S \Box\top\leftrightarrow\top$ and $\vdash_S (\top\to B)\leftrightarrow B$; hence

$$\vdash_S \Box A\to\Box\Box A$$
$$\vdash_S \Box\Box A\to\Box A.$$

Since S is clodsed under substition, this must hold for any A.

KD4! satisfies the closure condition for S by Lemma 2 (or by closre under RNc.). Hence, S is at least as strong as KD4!, and it follows from Theorem 1 that it need not be stronger.

3.2 We proceed as for 3.1, having in addition that since

$$\vdash_C A\vee\neg A,$$

where A is atomic, we must have

$$\vdash_S \Box(\Box A\vee\Box\neg\Box A), \qquad \text{i.e.}$$
$$\vdash_S \Box((\Box A\to\bot)\to\Box(\Box A\to\bot)),$$

which by the closure condition for S, and closure under substitution, gives 5. To show that KD45 satisfies the closure condition for S we use Lemma 2.

Q.E.D.

Alternatively, the results of Theorem 3 ould be expressed by saying that KD4! and KD45 are the minimal normal modal systems in which H and C, respectively, can be embedded by the translation which prefixes $\Box$ to every *proper* subformula. This is essentially Gödel's translation (v. Gödel 1933, McKinsey & Tarski 1948 and Czermak 1975).

In an analogous way we can show the following

THEOREM 4. 4.1 S4 *is the minimal normal modal system having all the formulae* $\Box A \to A$ *as theorems, in which* H *can be embedded by* t_1.

4.2. S5 *is the minimal normal modal system having all the formulae* $\Box A \to A$ *as theorems, in which* C *cen be embedded by* t_1.

Let K$\Box$(D!) and K$\Box$(D45) be obtained from KD4! and KD45, respectively, by taking as axiom-schemata

$$\begin{array}{ll} \Box D. & \Box(\Box A \to \Diamond A) \\ \Box 4. & \Box(\Box A \to \Box\Box A) \\ \Box 4c. & \Box(\Box\Box A \to \Box A) \\ \Box 5. & \Box((\Box A \to \bot) \to \Box(\Box A \to \bot)) \end{array}$$

instead of D., 4., 4c. and 5., respectively. That these two systems are properly contained in KD4! and KD45, respectively, is shown by interpreting $\Box A$ as $\top$, for every A. Then all the theorems of these two systems are tautologies, but $\Box \bot \to \bot$ is not a tautology with this interpretation; hence, D. is not a theorem of either of these systems.

For these systems we can shown the following

THEOREM 5. 5.1 K$\Box$(D4!) *is the minimal normal modal system in which* H *can be embedded by* t_1.

5.2. K$\Box$(D45) *is the minimal normal modal system in which* C *can be embedded by* t_1.

Demonstration: 5.1 First we show by an easy induction on the length of proof that if $\vdash_{KD4!} B$ then $\vdash_{K\Box(D4!)} \Box B$. Now suppose that $\vdash_{KD4!} t_1(A)$. $t_1(A)$ must be $\Box B$, for some B. Then we known that $\vdash_{KD4!} B$, by Lemma 2 (or closure under R Nc.). As we have shown above, it follows that $\vdash_{K\Box(D4!)} \Box B$. The converse being trivial, we have

$$\vdash_{KD4!} t_1(A) \qquad \text{iff} \qquad \vdash_{K\Box(D4!)} t_1(A),$$

Then consider the demonstration of Theorem 3.1 and note that in the minimal normal modal system satisfying the condition of 5.1 we must have $\Box D.$, $\Box 4.$ and $\Box$ 4c.

For 5.2 we proceed analogously. Q.E.D.

Let KD4b and K$\Box$(D4b) be obtained by replacing 4c. and $\Box$ 4c. in KD4! and K$\Box$(D4!) by, respectively, (b') and

$$\Box\,(b') \qquad \Box(\Box(\Box\, A \vee \Box\, B) \rightarrow (\Box\, A \vee \Box\, B)).$$

It can be shown that K$\Box$(D4b) is properly contained in KD4b, which is properly contained in S4.

For these systems we can show the following theorem, with which we conclude this paper:

THEOREM 6. 6.1 KD4b *is the minimal normal modal system closed under* $\frac{\Box\, A}{A}$, *where* $\Box\, A$ *is* $t_2(B)$ *for some* $\vdash_{\overline{H}} B$, *in which* H *can be embedded by* t_2.

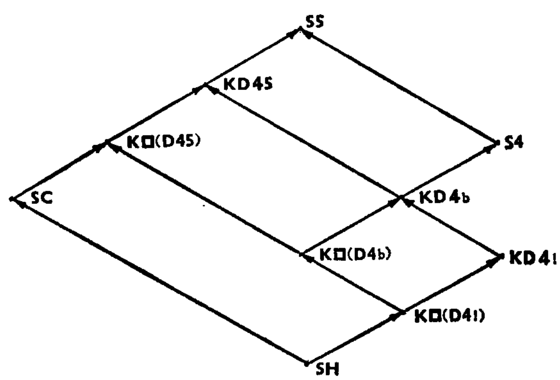
6.2 K$\Box$(D4b) *is the minimal normal modal system in which* H *can be embedded by* t_2.

Demonstration: 6.1 It is easily shown that H can be embedded in KD4b by t_2. That (b') is essential we have shown in the remarks after Theorem 1. That KD4b satisfies the closure condition can be shown by demonstrating closure under RNc.

6.2 We first shown that $\vdash_{\overline{KD4b}} t_2(a)$ iff $\vdash_{\overline{K\Box(D4b)}} t_2(A)$. From left to right we make an induction on the complexity of $t_2(A)$, using the facts: $\vdash_{\overline{KD4b}} B \wedge C$ iff $\vdash_{\overline{KD4b}} B$ and $\vdash_{\overline{KD4b}} C$; $\vdash_{\overline{KD4b}} B \vee C$ iff either $\vdash_{\overline{KD4b}} B$ dor $\vdash_{\overline{KD4b}} C$, where $B \vee C = t_2(D)$ for some D; and if $\vdash_{\overline{KD4b}} B$, then $\vdash_{\overline{K\Box(D4b)}} \Box\, B$. The rest follows easily.

Q.E.D.

On the following diagram we display the interrelations of the modal systems mentioned ion this paper (arrows indicate proper inclusion):



The results presented here can be extended to the corresponding predicate calculi*

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