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AN UPPER ESTIMATION FOR THE EIGENFREQUENCES OF VIBRATING LIAPUNOFF BODIES (FIRST BOUNDARY VALUE PROBLEM)

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1. For any bounded (open) domain $\Omega \subset R^n$ and for $j = 1, 2, \cdots$ we define the j-th cigenfrequency $\lambda_j(\Omega)$ of the homogeneous Ω shaped and at its boundary $\partial\Omega$ fixed vibrating body by

$$\Lambda_j(\Omega) = \inf_{L \in M_j} \sup_{f \in L} \left(\int_\Omega \left\| \operatorname{grad} f(x) \right\|^2 dx \Big/ \int_\Omega |f(x)|^2\, dx \right)^{1/2} \tag{1}$$

where M_j denotes the collection of the j-dimensional subspaces of the Soboleff space $W_0^{1,2}$.*

As it is well-known (*cf.* [**1**]), if $\partial\Omega$ is an $(n-1)$-dimensional C^2-submanifold of R^n, then the eigenvalues of the boundary value problem

$$\Delta f + \Lambda^2 \cdot f = 0, \quad f \in C_0^\infty(\overline{\Omega})$$

are given by (1). On the other hand, it is also shown (e.g. [**1, 3**]) that all the mappings $\Omega \mapsto \Lambda_j(\Omega)$ are continuous with respect to the topology on the set of the bounded R^n-domains defined by the usual Hausdorff distance.

While for all dimensions it is clairified (cf. [**2**]) that

$$\Lambda_j(\Omega) \geq \Lambda_j\left(\left\{x \in R^n : \|x\| < \left(\frac{\mathrm{vol}_n \Omega}{\omega_n}\right)^{1/n}\right\}\right) \qquad (j = 1, 2, \dots)$$

where vol_n denotes the n-dimensional Hausdorff measure and $\omega_n \equiv \mathrm{vol}_n\{x \in R^n : \|x\| < 1\}$, it is not at all known over two dimensions what kind of effective upper estimates can be given for the value of $\Lambda_j(\Omega)$ depending on some geometric parameters of Ω. However, for convex Ω-s it was proved (cf. [**3**]) that the analogous

*i.e. $f \in W_0^{1,2}$ if grad f exists in the weak sense and belongs to $L^2(\Omega)$ and supp f is contained in some copmact set which does not meet $\partial\Omega$.

of the best known two dimensional estimates (see [**4**]) hold in general (and can not be improved). The purpose of this paper is to extend a theorem of G. PÓLYA [**5**] concerning convex Ω-s to a larger class of geometrical figures (for generalized Liapunoff bodies, defined in the next sections).

2. A bounded domain $\Omega(\subset R^n)$ whose boundary is an $(n-1)$-dimensional C^2-submanifold in R^n is called a *Liapunoff body* if the Minkowskian curvature of $\partial\Omega$ with respect to the outward from Ω oriented normal vectors is non-negative at any point of $\partial\Omega$. (Remark here that the convexity of Ω is equivalent to the non-negativeness of the main curvatures of $\partial\Omega$ separately).

According to some recent results in geometric measure theory, it is possible to give a generalization of the concept of Minkowskian curvature which applies to the boundary of any open subset $\Omega \subset R^n$. This can be carried out as follows:

It is shown in [**6**, Theorem 5] that by setting

(2)
$$K \equiv \{(x,k) : x \in \partial\Omega,\ \|k\| = 1,\ \exists\varrho > 0\ x + \varrho k \in \Omega \text{ and dist } (x + \varrho k, \partial\Omega) = \varrho\}$$

(3)
$$h(x,k) \equiv \sup\{\xi > 0 : \operatorname{dist}(x + \varrho k,\ \partial\Omega) = \varrho,\ \forall\varrho \in [0,\xi]\} \quad \text{for } (x,k) \in K,$$

one always can find a σ-finite Borel measure μ on K and Borel measurable functions $a_j : K \to R\,(j = 0, \dots, n-1)$ such that for all $f \in L^1(\Omega)$ we have

(4)
$$\int\limits_{\Omega} f(y)\,dy = \int\limits_{K} \int\limits_{0}^{h(x,k)} f(x+\varrho k) \sum_{j=0}^{n-1} a_j(x,k)\varrho^j \,d\varrho\,d\mu(x,k).$$

Here $d\mu$ and $a_0, \dots, a_{n-1}$ are necessarily determined only up to the signed measures

(5)
$$d\,\alpha_j \equiv a_j\,d\mu \qquad (j = 0, \dots, n-1)$$

in the sense that if (4) is satisfied when $d\,\mu$ and $a_0, \dots a_{n-1}$ are replaced $dy\ d\tilde{\mu}$ and $\tilde{a}_0, \dots, \tilde{a}_{n-1}$, respectively, then we have

$$\int\limits_{E} a_j\,d\mu = \int\limits_{E} \tilde{a}_j\,d\tilde{\mu} \qquad (j = 0, \dots, n-1)$$

for all such $E \subset K$ that $\int\limits_E a_j\,d\mu$ or $\int\limits_E \tilde{a}_j\,d\tilde{\mu}$ makes sense. Thus, for $d\,\mu$ (and hence also $\delta\,\tilde{\mu}$)-almost every $(x,k) \in K$, the polynomials $\varrho \mapsto \sum\limits_{j-0}^{n-1} a_j(x,k)\varrho^j$ and $\varrho \mapsto \sum\limits_{j=0}^{n-1} \tilde{a}_j(x,k)\varrho^j$ differ only in a positive constant factor.

We shall call the measure $d\,\alpha_j$ defined by (5), which depends only on the geometric parametars of Ω, the *j-th curvature measure of the boundary of* Ω. This terminology is motivated by the relation (6) below. The formula (4) can be considered as a generalization of the main theorem in [**11**].

In the classical case, when $\partial\Omega$ is C^2-smooth, we have $(x,k) \in K$ if and only if k is the toward Ω oriented normal vector (of unit length) of the surface $\partial\Omega$ at the point $x(\in \partial\Omega)$. Now there is a natural choice for $d\mu$ and $a_0, \dots a_{n-1}$: We can define $d\mu$ by

$$\mu(E) = \mathrm{vol}_{n-1}\{x \in \partial\Omega : \exists k \ (x,k) \in E\}$$

(for the Borel measurable subsets E of K; vol_{n-1} denoting the $(n-1)$-dimensional Hausdorff measure). Then $a_0(x,k), \dots, a_{n-1}(x,k)$ are the coefficients of the polynomial

$$\varrho \mapsto \sum_{j=0}^{n-1} a_j(x,k)\varrho^j \equiv \prod_{i-1}^{n-1}(1-\varrho k_i(x)) \tag{6}$$

where $k_0(x), \dots, k_{n-1}(x)$ denote the main curvatures with respect to the outer normal of $\partial\Omega$ at the point x. Thus, in this special case, the curvature measures $d\,\alpha_j$ (defined by (5) and (6)) are all absolutely continuous with respect to $d\,\alpha_0$ and the Minkowskian curvature $k_1 + \dots + k_{n-1}$ of $\partial\Omega$ coincides with $-\frac{d\,\alpha_1}{d\,\alpha_0}$. Therefore, to save the most properties of the classiacal case, we define generalized Liapunoff bodies in the following way:

Definition. A bounded domain Ω in R^n is said to be a *generalized Liapunoff body* if all its cutvature measures $\alpha_j (j = 0, \dots, n-1)$ introduced above are absolutely continuous with respect to α_0 and the function $-\frac{d\,\alpha_1}{d\,\alpha_0}$ (which we shall call now the Minkowskian curvature of $\partial\Omega$) is non-negative.

THEOREM 1. *If $\Omega \in R^n$ is a generalized Liapunoff body then the function $\varrho \mapsto \mathrm{vol}_{n-1}\partial(\Omega_{-\varrho})$ (where $\Omega_{-\varrho}$ denotes the inner parallel domain of radius $\varrho > 0$ of Ω, i.e. $\Omega_{-\varrho} \equiv \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varrho\}$) is non-increasing for $0 < \varrho < \infty$.*

Proof. Let Ω denote a generalized Liapunof body and define K and h as in (2) and (3). Choose $d\,\mu$, $a_0, \dots, a_{n-1}$ so that (4) be satisfied. It is proved in [**6** Theorem 5, Corollary] that here we necessarily have

$$\sum_{j=0}^{n-1} a_j(x,k)\varrho^j > 0 \text{ whenever } 0 < \varrho < h(x,k) \qquad ((x,k) \in K). \tag{7}$$

Remark that (7) is not a simple corollary of (4) and the positiveness of the operation $f \mapsto \int_\Omega f(y)\,dy$ because these facts ensure only $\sum_{j=0}^{n-1} a_j(x,k)\varrho^j \geq 0$ for $0 < \varrho \leq h(x,k)$. It is easy to see from (2) and (3) that $\Omega_{-\varrho} = \{x + \xi k : (x,k) \in K$ and $\varrho < \xi \leq h(x,k)\}$ and hence

$$1_{\Omega-\varrho}(x + \xi k = 1_{[\varrho, h(x,k)]}(\xi) \text{ for } (x,k) \in K \text{ and } \xi \in [0, h(x,k)]. \tag{8}$$

From (4) and (8) we obtain

$$\mathrm{vol}_n\Omega_{-\varrho}\int\limits_K\int\limits_\varrho^\infty \varphi_{x,k}(\xi)\,d\xi\,d\mu(x,k) \tag{9}$$

where $\varphi_{x,k}(\xi)=\sum\limits_{j=0}^{n-1}a_j(x,k)\xi^j\cdot 1_{[0,k(x,k)]}(\xi)$.

Recall that for μ-almost every $(x,k)\in K$, the polynomial $P_{x,k}:\xi\mapsto\sum\limits_{j=0}^{n-1}a_j(x,k)\xi^j$ has only real roots (cf. [**6**, Theorem 5]) and that from the definition of Liapunoff bodies and (7) we have $P_{x,k}(0)>0$ and $P'_{x,k}(0)=a_1(x,k)=\frac{d\alpha_1}{d\alpha_0}\Big|_{(x,k)}\le 0$ (for μ-almost every $(x,k)\in K$).

Since, in general, a polynomial $P:R\mapsto R$ having only real roots and such that $P(0)>0$ and $P'(0)\le 0$ is constant or has positive root and P desreases on $[0,\min\{\xi>0:P(\xi)=0\}]$ (cf. [10, Lemma]) it follows from (7) and the definiton of $P_{x,k}$ that the functions $\varphi_{x,k}$ are monotone decreasing on the whole $[0,\infty)$ for μ-almost all $(x,k)\in K$. Therefore, from (9) we deduce that the function

$$\varrho\mapsto-\frac{1}{2}\left(\frac{d^+}{d\xi}\Big|_\varrho+\frac{d^-}{d\xi}\Big|_\varrho\right)\mathrm{vol}_n\Omega_{-\xi}$$

is well-defined for all $\varrho>0$ and it is decreasing.

However, it is shown in [**7**] that the $(n-1)$-dimensional Minkowski content of $\partial(\Omega_{-\varrho})$ equals to $-\frac{1}{2}\left(\frac{d^+}{d\xi}\Big|_\varrho+\frac{d^-}{d\xi}\Big|_\varrho\right)\mathrm{vol}_n\Omega_{-\xi}$. Since the boundary of any bounded parallel set is easily an $(n-1)$-rectifiable subset of R^n (for definitions see [**8**]), a well-known theorem of M. Kneser (cf. [**8**]) implies that $\mathrm{vol}_{n-1}\partial\Omega_{-\varrho}=(n-1)$-Minkovski content $(\partial(\Omega_{-\varrho}))=\frac{1}{2}\left(\frac{d^+}{d\xi}\Big|_\varrho+\frac{d^-}{d\xi}\Big|_\varrho\right)\mathrm{vol}_n\Omega_{-\xi}$. This completes the proof.

3. The following geometric estimation is given in [**10**] for the eigenfrequences $\Lambda_j(\Omega)$:

Theorem 2. *Let Ω be such a bounded in R^n that $\sup\limits_{\varrho>0}\mathrm{vol}_{n-1}\partial(\Omega_{-\varrho})<\infty$. Then, by setting $l(\Omega)\equiv\mathrm{vol}_n\Omega/\sup\limits_{\varrho>0}\mathrm{vol}_{n-1}\partial(\Omega_{-\varrho})$, we have*

$$\Lambda_1(\Omega)^2\le\frac{\pi}{2}\cdot l(\Omega)^{-1}$$

The ideas of the proof of Theorem 2 are essentially based upon those of the article [**5**].

Thus Theorem 1 directly yields our chief observation

THEOREM 3. *If Ω is a generalized Liapunoff body in R^n then*

$$\Lambda_1(\Omega)^2 \leq \frac{\pi}{2} \cdot \frac{\lim_{\varrho \downarrow 0} \mathrm{vol}_{n-1} \partial(\Omega_{-\varrho})}{\mathrm{vol}_n \Omega} \tag{10}$$

In particular, if $\partial\Omega$ is a C^2-smooth hypersurface, then

$$\Lambda_1(\Omega)^2 \leq \frac{\pi}{2} \cdot \frac{\mathrm{vol}_{n-1} \partial\Omega}{\mathrm{vol}_n \Omega}$$

Remark. One can prove that for any generalized Liapunoff body $\Omega \subset R^n$ we have on the right hand side of (10)

$$\lim_{\varrho \downarrow 0} \mathrm{vol}_{n-1} \partial(\Omega_{-\varrho}) = \int_{\partial\Omega} \text{cardinality } \{k : (x,k) \in K\} d\,\mathrm{vol}_{n-1}(x).$$

Proof. By [**6**, Lemma 9] we can fix disjoint Borel subsets $B_1, B_2, \ldots$ of K and open sets $\Omega^{(1)}, \Omega^{(2)}, \ldots \subset R^n$ with positive reach (for def. see [**6**] or [**11**]) such that by setting $\varrho_m \stackrel{\text{def}}{=} \inf\left\{ \frac{1}{2}\text{reach } \Omega^{(m)},\ h(x,k) : (x,k) \in B_m \right\}$ and $K_m \stackrel{\text{def}}{=} \{(x,k) : x \in \partial\Omega^{(m)},\ \|k\| = 1,\ \exists \varrho > 0\, x + \varrho k \in \Omega^{(m)},\ \mathrm{dist}(x, \partial\Omega^{(m)}) = \varrho\}$ we have

$$K = \bigcup_{m=1}^{\infty} B_m, \ \ \varrho_m > 0 \quad \text{and} \quad B_m \subset K_m \quad (m = 1, 2, \ldots).$$

Using [**6**, Theorem A, B] we can see that for each point $y \in \partial(\Omega^{(m)}_{\varrho_m})$ there exists a unique pair $(x_m(y),\ k_m(y))$ in K_m with the property $y = x_m(y) + \varrho_m k_m(y)$ and, by [**8**, 3.2.3], for any fixed $\xi \in R$, the mapping $T^m_\xi : y \mapsto y + \xi k(y)$ satisfies

$$\int_{T^m_\xi(S)} \mathrm{card}(T^m_\xi)^{-1}(z) \mathrm{vol}_{n-1} z = \int_S [1 + (\xi - \varrho_m) k^m_1(y)] \cdots [1 + (\xi - \varrho_m) k_{n-1}(y)] d\mathrm{vol}_{n-1}(y),$$

where $k^m_1, \ldots, k^m_{n-1}$ are the main curvatuires of $\partial(\Omega^{(m)}_{-\varrho_m})$ (cf. [**6**, Theorem B]) defined vol_{n-1} almost everywhere on $\partial(\Omega^{(m)}_{-\varrho m})$. Hence in particular,

$$\int_{\{x : \exists k (x,k) \in B_m\}} \mathrm{card}\{k : (x,k) \in B_m\} d\mathrm{vol}_{k-1}(x) = \int_{\{x + \varrho_m k : (x,k) \in B_m\}} [1 - \varrho_m k^m_1(y)] \cdots [1 - \varrho_m k^m_{n-1}(y)] d\mathrm{vol}_{n-1}(y). \tag{11}$$

The proof of the main Theorem in [**6**] shows (cf. [**6**, (5'), (5")]) that the measures $a_j d\mu$ in formula (5) are given by

$$\sum_{j=0}^{n-1} \varrho^j \int_B a_j d\mu = \int_{\{x + \varrho_m k (x,k) \in B\}} [1 + (\varrho - \varrho_m) k^m_1(y)] \cdots [1 + (\varrho - \varrho_m) k^m_{n-1}(y)] d\mathrm{vol}_{n-1}(y)$$

for $B \subset B_m$ and $\varrho \in R (m = 1, 2, \dots)$. Thus (11) yields

$$\int_{\partial\Omega} \operatorname{card}\{k : (x,k) \in K\} d\mathrm{vol}_{n-1}(x) = \int_K a_0(x,k) d\mu(x,k). \tag{12}$$

On the other hand, applying the functions $\varphi_{x,k}((x,k) \in K)$ introduced in formula (9), we see

$$\int_K a_0 d\mu = \int_K \lim_{\varrho \downarrow 0} \varphi_{x,k}(\varrho) d\mu(x,k) = \lim_{\varrho \downarrow 0} \int_K \varphi_{x,k}(\varrho) d\mu(x,k)$$

since the functions $\varphi_{x,k}$ are monotone decreasing for all fixed $(x,k) \in K$. Now, to complete the proof, we need only to remark that, by (9) and by [**8**, 3.2.34], we have

$$\mathrm{vol}_{n-1}\partial\Omega_{-\varrho} = -\frac{d}{d\varrho}\mathrm{vol}_n\Omega_{-\varrho} = -\frac{d}{d\varrho}\int_k \int_\varrho^\infty \varphi_{x,k}(\xi) d\xi d\mu(x,k) = \int_K \varphi_{x,k}(\varrho) d\mu(x,k)$$

for almost every $\varrho \in (0, \infty)$.

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