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FUNCTIONAL EQUATIONS OF GENERALIZED ASSOCIATIVITY, BISYMMETRY, TRANSITIVITY AND DISTRIBUTIVITY

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A. Krapež

In this paper we solve functional equations of generalized associativity (1), bisymmetry (2), transitivity (3) and distributivity (4).

(1) $$A(x, B(y, z)) = C(D(x, y), z)$$

(2) $$A(B(x, y), C(u, v)) = D(E(x, u), F(y, v))$$

(3) $$A(B(x, y), C(y, z)) = D(x, z)$$

(4) $$A(x, B(y, z)) = C(D(x, y), E(x, z))$$

We say that functional equation is generalized if in this equation every unknown function occurs exactly once. We assume that unknown functions are grupoids, defined on some nonempty set S.

Despite numerous literature (see for example [**1**]) about the subject, these equations are not solved under such weak conditions.

Here we continue our work, started in [**2**], on various generalized functionel equations on grupoids.

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Following common practice, for a given function $P : S \to S'$, we define an equivalence relation $kerP$ on S as

$$x \, ker \, P y \quad \text{iff} \quad Px = Py$$

Bijection $f_p : S/_{ker\,P} \to P(S)$, defined by $f_p(x^{ker\,P}) = Px$, is naturally connected with P.

If an equivalence α on S^2 is given, then:

$$(a, b, c) \, \alpha_1 \, (p, q, r) \quad \text{iff} \quad a = p \quad (b, c)\alpha(q, r)$$
$$(a, b, c) \, \alpha_3 \, (p, q, r) \quad \text{iff} \quad (a, b)\alpha(p, q) \ \text{ and } \ c = r$$

If β is also an equivalence on S^2, then:

$$(a,b,c,d)\,\alpha\Delta_1\beta\,(p,q,r,s) \quad \text{iff} \quad (a,b)\alpha(p,q) \quad \text{and} \quad (c,d)\beta(r,s)$$
$$(a,b,c,d)\,\alpha\Delta_2\beta\,(p,q,r,s) \quad \text{iff} \quad (a,c)\alpha(p,r) \quad \text{and} \quad (b,d)\beta(q,s)$$
$$(a,b,c)\,\alpha\Delta_3\beta\,(p,q,r) \quad \text{iff} \quad (a,b)\alpha(p,q) \quad \text{and} \quad (b,d)\beta(q,r)$$
$$(a,b,c)\,\alpha\Delta_4\beta\,(p,q,r) \quad \text{iff} \quad (a,b)\alpha(p,q) \quad \text{and} \quad (a,c)\beta(p,r)$$

If $\alpha_1,\dots,\alpha_n$ are equivalences on S^2, then:

$$(a_1,\dots,a_{n+1})\Delta(\alpha_1,\dots,\alpha_n)(b_1,\dots,b_{n+1}) \quad \text{iff}$$
$$\text{for all } k=1,\dots,n \ \ (a_k,a_{k+1})\alpha_k(b_k,b_{k+1})$$

Also let:

$$(a,b,c)\delta(p,q,r) \quad \text{iff} \quad a=p \quad \text{and} \quad c=r$$
$$(a_1,\dots,a_{n+1})\gamma(b_1,\dots,b_{n+1}) \quad \text{iff} \quad a_1=b_1 \quad \text{and} \quad a_{n+1}=b_{n+1}$$

Suppose $\alpha,\beta,\alpha_1,\dots,\alpha_n$ are equivalences on S^2, τ and π are equivalences on $S^3(S^4)$ and σ is an equivalence on S^{n+1} for some $n\in N$. The following functions are not always well defined but we can easily formulate conditions under which they are. Whenever we use these functions we will prove that they are well defined.

$$\Gamma_A : (x,(y,z)^\alpha) \mapsto (x,y,z)^\tau$$
$$\Gamma_C : ((x,y)^\alpha,z) \mapsto (x,y,z)^\tau$$
$$\Gamma_1 : ((x,y)^\alpha,(u,v)^\beta \mapsto (x,y,u,v)^\pi$$
$$\Gamma_2 : ((x,u)^\alpha,(y,v)^\beta) \mapsto (x,y,u,v)^\pi$$
$$\Gamma_3 : ((x,y)^\alpha,(y,z)^\beta) \mapsto (x,y,z)^\tau$$
$$\Gamma_4 : (x,y) \mapsto (x,a,y)^\tau \quad \text{for some} \quad a\in S$$
$$\Gamma_5 : ((x,y)^\alpha,(x,z)^\beta) \mapsto (x,y,z)^\tau$$
$$\Gamma_6 : ((x_1,x_2)^{\alpha_1},\dots,(x_n,x_{n+1})^{\alpha_n}) \mapsto (x_1,\dots,x_{n+1})^\sigma$$
$$\Gamma_7 : (x,y) \mapsto (x,a_2,\dots,a_n,y)^\sigma \quad \text{for some} \quad a_2,\dots,a_n\in S$$

The functional equation of generalized associativity (1) is solved in [**2**] (Th 5). The solution given here (Th 1) is different but similar to that in [**2**], so we do not give the proof here. The present solution has some advantages, it is simpler and similar in form to solutions of other equations.

THEOREM 1. *The general solution of the generalized associativity equation* (1), *is given by:*

B and D are arbitrary grupoids on S

$$A(x,y)=\begin{cases} f_T\Gamma_A(x,f_B^{-1}y) & \text{for} \quad (x,y)\in S\times B(S,S)\\ \text{arbitrary} & \text{otherwise}\end{cases}$$
(5)
$$C(x,y)=\begin{cases} f_T\Gamma_C(f_D^{-1}x,y) & \text{for} \quad (x,y)\in D(S,S)\times S\\ \text{arbitrary} & \text{otherwise}\end{cases}$$

where T *is an arbitrary* 3*-grupoid such that:*

$$(ker\,B)_1 \vee (ker\,D)_3 \subset ker\,T$$

THEOREM 2. *The general solution of the generalized bisymmetry equation* (2) *is given by:*

$$B, C, E \text{ and } F \text{ are arbitrary grupoids on } S$$

$$(6)\qquad \begin{aligned} A(x,y) &= \begin{cases} f_T\Gamma_1(f_B^{-1}x, f_C^{-1}y) & \text{for} \quad (x,y)\in B(S,S)\times C(S,S) \\ \text{arbitrary} & \text{otherwise} \end{cases} \\ D(x,y) &= \begin{cases} f_T\Gamma_2(f_E^{-1}x, f_F^{-1}y) & \text{for} \quad (x,y)\in E(S,S)\times F(S,S) \\ \text{arbitrary} & \text{otherwise} \end{cases} \end{aligned}$$

where T *is an arbitrary* 4*-grupoid such that:*

$$(ker\,B\,\Delta_1\,ker\,C)\vee(ker\,E\Delta_2\,ker\,F)\subset ker\,T$$

Proof: (i) It can be easily checked that for given B, C, E, F and T, sixtuple (A, B, C, D, E, F), satisfying (6) is a solution of (2).

(ii) Let A, B, C, D, E, F be grupoids satisfying (2) and let:

$$T(x,y,u,v) = A(B(x,y), C(u,v))$$

Then from $(a,b,c,d)(ker\,B\Delta_1\,ker\,C)(p,q,r,s)$ it follows that $(a,b)ker\,B(p,q)$ (i.e.$B(a,b) = B(p,q)$) and $(c,d)\,ker\,C(r,s)$ (i e.$C(c,d) = C(r,s)$).

It follows that $A(B(a,b),\ C(c,d)) = A(B(p,q),\ C(r,s))$ i.e. $(a,b,c,d)\ ker\,T$ (p,q,r,s).

Similarly, from (a,b,c,d) $(ker\,E\Delta_2\,ker\,F)$ (p,q,r,s) it follows that (a,b,c,d) $ker\,T(p,q,r,s)$. This also means that Γ_1 and Γ_2 are well defined.

Let $x \in B(S,S)$ and $y \in C(S,S)$. Then there are z,u,v,w such that $x = B(z,u)$ and $y = C(v,w)$ and $A(x,y = A(B(z,u),\ C(v,w)) = f_T((z,u,v,w)^{ker\,T}) = f_T\Gamma_1((z,u)^{ker\,B},(v,w)^{ker\,C}) = f_T\Gamma_1(f_B^{-1}B(z,u), f_C^{-1}C(v,w)) = f_T\Gamma_1(f_B^{-1}x, f_C^{-1}y)$.

Analogously we can prove $D(x,y) = f_T\Gamma_2(f_E^{-1}x, f_F^{-1}y)$.

In [**2**] (example 1) it is proved that we can obtain the general solution of the generalized associativity on quasigroups, from the general solution in the grupoid case. We can prove that the same holds for the generalized bisymmetry (see also [**3**]).

THEOREM 3. *The general solution on quasigroups, of the generalized bisymmetry equation, is given by:*

$$\begin{aligned} A\,(x,y) &= A_1x + A_2y \\ B\,(x,y) &= A_1^{-1}(A_1B_1x + A_1B_2y) \\ C\,(x,y) &= A_2^{-1}(A_2C_1x + A_2C_2y) \\ D\,(x,y) &= D_1x + D_2y \\ E\,(x,y) &= D_1^{-1}(D_1E_1x + D_1E_2y) \\ F\,(x,y) &= D_2^{-1}(D_2F_1x + D_2F_2y) \end{aligned}$$

where $+$ *is an arbitrary abelian group and* $A_1, A_2, \dots, F_1, F_2$ *arbitrary permutations such that:*

$$\begin{aligned} A_1 B_1 &= D_1 E_1 \\ A_1 B_2 &= D_2 F_1 \\ A_2 C_1 &= D_1 E_2 \\ A_2 C_2 &= D_2 F_2 \end{aligned}$$

THEOREM 4. *The general solution of the generalized transitivity equation* (3) *is given by:*

$$B \text{ and } C \text{ are arbitrary grupoids on } S$$

$$(7) \qquad A(x,y) = \begin{cases} f_T \Gamma_3 (f_B^{-1} x, f_C^{-1} y) & \text{for} \quad (x,y) \in \bigcup_{z \in S} (B(S,z) \times C(z,S)) \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

$$D(x,y) = f_T \Gamma_4 (x,y)$$

where T *is an arbirary* 3*-grupoid such that:*

$$(ker\, B \Delta_3\, ker\, C) \vee \delta \subset ker\, T$$

Proof: (i) For given B, C and T quadruple (A, B, C, D) satisfying (7) is a solution of (3).

(ii) Let A, B, C, D be grupoids satisfying (3) and let

$$T(x,y,z) = A(B(x,y), C(y,z))$$

Then from $(a,b,c)(ker\, B \Delta_3 ker\, C)(p,q,r)$ it follows that $(a,b) ker\, B(p,q)$ (i.e. $B(a,b) = B(p,q)$) and $(b,c) ker\, C(q,r)$ (i.e. $C(b,c) = C(q,r)$). It follows that $A(B(a,b), C(b,c)) = A(B(p,q)), C(q,r))$ i.e. $(a,b,c) ker\, T(p,q,r)$.

Also, from $(a,b,c)\, \delta\, (p,q,r)$ it follows that $a = p$ and $c = r$. So $D(a,c) = D(p,r)$.and by (3), $A(B(a,b), C(b,c)) = A(B(p,q), C(q,r))$ i.e. $(a,b,c)\, ker\, T\, (p,q,r)$.

Consequently Γ_3 and Γ_4 are well defined.

Let $x \in B(S,z)$ and $y \in C(z,S)$ for some $z \in S$. Then there are $u, v \in S$ such that $x = B(u,z)$ and $y = C(z,v)$. Also:

$$A(x,y) = A(B(u,z), C(z,v)) = T(u,v,z) = f_T((u,z,v)^{ker\, T}) =$$

$$= f_T \Gamma_3 ((u,z)^{ker\, B}, (z,v)^{ker\, C}) = f_T \Gamma_3 (f_B^{-1} B(u,z), f_C^{-1} C(z,v)) = f_T \Gamma_3 (f_B^{-1} x, f_C^{-1} y)$$

$$D(x,y) = A(B(x,z), C(z,y)) = T(x,z,y) = f_T((x,z,y)^{ker\, T}) = f_T \Gamma_4 (x,y)$$

As in the case of associativity and bisymmetry equations, from the general solution of generalized transitivity equation in grupoid case, we can obtain the general solution in the quasigroup case (see [**3**]).

THEOREM 5. *The general solution on quasigroups, of the generalized transitivity equation is given by:*

$$\begin{aligned}
A(x,y) &= A_1x \cdot A_2y \\
B(x,y) &= A_1^{-1}(A_1B_1x \cdot A_1B_2y) \\
C(x,y) &= A_2^{-1}(A_2C_1x \cdot A_2C_2y) \\
D(x,y) &= D_1x \cdot D_2y
\end{aligned}$$

where $\cdot$ *is an arbitrary group (with an unit* e*) and* $A_1, A_2, \dots, D_1, D_2$ *arbitrary permutations such that:*

$$\begin{aligned}
A_1B_1 &= D_1 \\
A_1B_2x \cdot A_2C_1x &= e \\
A_2C_2 &= D_2
\end{aligned}$$

Contrary to generalized distributivity equation on quasigroups (see [**4**]), which is not solved, this equation in the grupoid case be solved easily.

THEOREM 6. *The general solution of the generalized distributivity equation is given by:*

$$\begin{aligned}
&B, D \text{ and } E \text{ are arbitrary on } S \\
&A(x,y) = \begin{cases} f_T\Gamma_A(x, f_B^{-1}) & \text{for} \quad (x,y) \in S \times (S,S) \\ \text{arbitrary} & \text{otherwise} \end{cases} \\
&C(x,y) = \begin{cases} f_T\Gamma_5(f_D^{-1}x, f_E^{-1}y) & \text{for} \quad (x,y) \in \bigcup\limits_{z \in S}(D(z,S) \times E(z,S)) \\ \text{arbitrary} & \text{otherwise} \end{cases}
\end{aligned} \tag{8}$$

where T *is an arbitrary* 3*-grupoid such that:*

$$(ker\, B)_1 \vee (ker\, D \Delta_4 ker\, E) \subset ker\, T$$

Proof: (i) For given B, D, E and T, quintuple (A, B, C, D, E) satisfying (8) is a solution of (4).

(ii) Let A, B, C, D, E be grupoids satisfying (4) and let:

$$T(x,y,z) = A(x, B(y,z))$$

Then from $(a,b,c)\,(ker\, B)_1\,(p,q,r)$ it follows that $a = p$ and $(b,c)\, ker\, B(q,r)$ (i.e. $B(b,c) = B(q,r)$). Consequently $A(a, B(b,c)) = A(p, B(q,r))$ i.e. (a,b, c) $ker\, T(p,q,r)$.

Also, from $(a,b,c)\,(ker\, D\Delta_4 ker\, E)\,(p,q,r)$ it follows that $(a,b) ker\, D(p,q)$ (i. e. $D(a,b) = D(p,q)$) and $(a,c) ker\, E(p,r)$ (i.e. $E(a,c) = E(p,r)$). It follows that $C(D(a,b),\, E(a,c)) = C(D(p,q),\, E(p,r))$ i.e. $A(a, B(b,c)) = A(p, B(q,r))$ which is the same as $(a,b,c)\, ker\, T(p,q,r)$.

Consequently Γ_A, Γ_5 are well defined.

Let $x \in S$ and $y \in B(S,S)$. Then there are $u, v \in S$ such that $y = B(u,v)$ and:

$$A(x,y) = A(x, B(u,v)) = T(x,u,v) =$$
$$= f_T\Gamma_A(x,(u,v)^{ker\,B}) = f_T\Gamma_A(x, f_B^{-1}B(u,v)) = f_T\Gamma_A(x, f_B^{-1}y).$$

If $x \in D(z,S)$ and $y \in E(z,S)$ for some $z \in S$, then there are $u, v \in S$ such that $x = D(z,u)$ and $y = E(z,v)$ and:

$$C(x,y) = C(D(z,u), E(z,v)) = A(z, B(u,v)) = T(z,u,v) =$$
$$= f_T((z,u,v)^{ker\,T}) = f_T\Gamma_5((z,u)^{ker\,D},(z,v)^{ker\,E}) =$$
$$f_T\Gamma_5(f_D^{-1}D(z,u), f_E^{-1}E(z,v)) = f_T\Gamma_5(f_D^{-1}x, f_E^{-1}y).$$

As an example that the above method of solving, can be extended to the case of n-ary grupoids, we give a general solution of so called generalized simple polytransitivity (see [**5**]):

$$A(B^1(x_1,x_2),\dots,B^n(x_n,x_{n+1})) = C(x_1,x_{n+1}) \tag{9}$$

THEOREM 7. *The general solution of the generalized simple polytransitivity* (9) *is given b:*

$B^1,\dots,B^n$ *are arbitrary grupoids on* S

$$A(x_1,\dots,x_n) = \begin{cases} f_T\Gamma_6(f_{B^1}^{-1}x_1,\dots,f_{B^n}^{-1}x_n) & \text{for} \quad (x_1,\dots,x_n) \in \partial_A \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

$C(x,y) = f_T\Gamma_7(x,y)$

where $\partial_A = \{(x_1,\dots,x_n) | \exists z_1,\dots,z_{n+1}(x_k = B^k(z_k,z_{k+1})$ *for all* $k = 1,\dots,n)\}$ *and* T *is an arbitrary* $n+1$*-grupoid such that:*

$$\Delta(ker\,B^1,\dots,ker\,B^n) \vee \gamma \subset ker\,T$$

Proof: Similar to the proof of Th 4.

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Matematički Institut
Knex Mihailova 35
11000 Beograd
Yugoslavia