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BAIRE OUTHER KERMELS OF SETS

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Abstract. The notion of an outher Baire kernel K of a set A is introduced as a Baire superset K of A such that the only Baire sets that are subsets of $K - A$ are of the first Baire category. Several results about outher Baire kernels are presented. These theorems are analogues of theorems of Abial dealing with the concept of the measurable outher kernel of a set.

1. Introduction. In this work every set considered will be a subset of the set R of all real numbers. Abian [**1**] in a work that will shortly appear considered the following definition.

Definition. Let A be a set and K a measurable set such that $A \subset K$ and such that the only measurable subsets of $K - A$ are of measure zero. If such a set K exists we call it a measurable outer kernel of A.

A set A is called a Baire set if it can be expressed in the form $A = (G-P) \cup Q$ where G is an open set and P, q are sets of the first Baire category (i.e. countable unions of nowhere dense sets). Analogues between measurable sets and Baire sets have been extensively studied in Oxtoby's book "Measure and Category" [**13**]. Much earlier Hausdorff [**4**] has commented on the connection between measurable sets and Baire sets. The current author has written several papers dealing at least in part with this connection ([**7, 8, 9,19,11,12**]).

In this work we consider an analogue of Abian's concept of a measurable outer kernel of a set.

Definition. A Baire set K is called an outer Baire kernel of a set A if and only if K is a superset of A and each Baire subset of $K - A$ is of the first Baire category.

Analogues of Abian's theorems will be proved.

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2. Results.

THEOREM 1. *Every set A has an outher Baire kernel.*

Proof. If A is a Baire set, then A possesses an outer Baire kernel, namely $K = A$ is an outer Baire kernel of A. In particular, if A is of the first Baire category than A has an outer Baire kernel. Suppose that A is an arbitrary set of the second Baire category. Let G denote a fixed open superset of A. Further let the sequence $(I_n)_{n=1}^{\infty}$ denote all closed intervals with rational endpoints. A descending sequence $(G)_{n=1}^{\infty}$ can be construced as follows.

If $A \cap I_1$ is a set of the first category, define

$G_1 = G - I_1$, whereas

if $A \cap I_1$ is a set of the second category, define

$G_1 = G$.

Using induction we can continue this process to obtain a decending sequence $(G_n)_{n=1}^{\infty}$ of open subsets of G such that

$G_n = G_{n-1} - I_n$ if $A \cap I_n$ is a set of the first Baire category and

$G_n = G_{n-1}$ if $A \cap I_n$ is a set of the second Baire category.

Set
$$K = \left(\bigcap_{n=1}^{\infty} G_n\right) \cup P, \qquad \text{where}$$

$P = \cup[A \cap I_n;\ A \cap I_n$ is of the first Baire category$]$. K is a Baire set (see [**13**], pp. 20) and K clearly is a superset of A.

We now show that K is an outer Baire kernel of A. To see this suppose that $Z = (\Gamma - P_1) \cup P_2$, with $Gamma$ open and $P_1,\ P_2$ sets of the first Baire category, is a subset of $K - A$. We will show that Z is of the first Baire category (i.e. $\Gamma = \varnothing$). If Z is of the second Baire category (i.e. $\Gamma \neq \varnothing$), then there exists a fixed positive integer m such that $I_m \subset \Gamma$.

Therefore

$$(*) \qquad I_m - P \subset Z \qquad \text{and} \qquad (I_m - P_1) \cap A = \varnothing$$

holds

This, in turn, implies that $I_m \cap A$ is of the Baire category and therefore $G_m = G_{m-1} - I_m$. The last equality implies

$$i_m \cap \bigcap_{n=1}^{\infty} G_n = \varnothing$$

and therefore

$(I_m - P_a) \cap K$ is of the first category which contradicts *), and hence the proof is complete. We now present a lemma which will be needed later.

Lemma. *If A is any set of real numbers, then $A-G$ is a set of the first Baire category, where G is the union of all open intervals with rational endpoints having the property that each of its subintervals has second category intersecstion with A.*

Proof. Suppose A is a set such A—G is of the second category. Let $(I_k;\ k \in N)$, (where N is the set of natural numbers), denote an enumeration of the collection of all open intervals with rational endpoints. Then G can be written in the form $G = \cup[I_k;\ k \in M]$ for some subset M of N, where $k \in M$ if and only if $J \cap A$ is a set of the second category for every subinterval J of I_k. If $k \in N - M$ there exists a subinterval J_k of I_k such that $A \cap J_k$ is of the first category. Therefore it follows that $T = (A - G) - \cup[J_k;\ k \in N - M]$ is a set of the second Baire category. But this is impossible since T is nowhere dense in R, as $T \cap I_k = \varnothing$ if $k \in M$ and $T \cap J_k = \varnothing$ if $k \in N - M$. This completes the proof of our lemma.

We now proceed with analogues of Abian's results.

Theorem 2. *If $K = (G - P) \cup Q$ is a superset of A, where G is the set given in the above lemma and P, Q are sets of the first Baire category, then K is an outer Baire kernel of A.*

Proof. Suppose $K = (G - P) \cup Q$ is a superset of A, where G is given as in our lemma. Suppose that $Z = (\Gamma - P_1) \cup P_2$ is a subset of $K - A$, where Γ is an open set and P_1, P_2 are sets of the first Baire category. We will show that Z is a set of the first Baire category, i.e. that $\Gamma = \varnothing$. Suppose Z is a set of the second Baire category, then there exists an open interval I such that $I \subset \Gamma$. It follows that $I \cap A$ is a set of the first Baire category. If J is any open interval with rational endpoints such that $I \cap J \neq \varnothing$, then it follows that $(I \cap J) \cap A$ is a set of the first Baire category and hence J is not one of the intervals in the collection of intervals whose union is G. Therefore $G \cap I\varnothing$, which contradicts the fact that $I = P_1 \subset Z \subset K$. This completes the proof of Theorem 2.

We now prove the converse of the last theorem.

Theorem 3. *If K is an outer Baire kernel of A then K can be written in the form $K = (G - P) \cup Q$, where G is given as in our lemma and P, Q are sets of the first Baire category.*

Proof. By our lemma $K' = G \cup (A - G)$ is a Baire superset of A. Therefore by Theorem 2, K' is an outer Baire kernel of A. Clearly $K = (K' - (K' - K)) \cup (K - K')$. Since $K' - K$ and $K - K'$ are Baire sets and from the definition of the outer Baire kernel of a set it follows that $K' - K$ and $K - K'$ are sets of the first Baire category. Therefore K can be written in the form $K = \{[G \cup (A - G)] - K' - K) \cup (K - K')$ and hence it follows that K can be written in the desired form, completing the proof of our theorem.

Combining Theorems 2 and 3, we have:

Theorem 4. *A Baire superset K of a set A is an outer Baire kernel of A if and only if K can be written in the form $K = (G - P) \cup Q$, where G is given as in our lemma and P, Q are sets of the first Baire category.*

Now we will show:

Theorem 5. *The union of the outer Baire kernels of a denumerable number of sets is an outer Baire kernel of their union.*

Proof. The proof of this theorem is an exact analogue of the proof of Theorem 5 in Abian's work, but is included for complereness. For every natural number i, let K_i be an outer Baire kernel of the set A_i. We show that $\bigcup_{i=1}^{\infty} K_i$ is an outer Baire kernel of $\bigcup_{i=1}^{\infty} A_i$. To see this $\Big($ as $\bigcup_{i=1}^{\infty} K_1$ is a Baire superset of $\bigcup_{i=1}^{\infty} A_i$, see [**13**], pg. 19$\Big)$ it is enough to prove that if Z is a Baire set and $Z \subseteq \left(\bigcup_{i=1}^{\infty} K_1 - \bigcup_{i=1}^{\infty} A_i\right)$, then Z is a set of the first Baire category. But in this case $Z \subseteq \bigcup_{i=1}^{\infty}\left(K_i - \bigcup_{j=1}^{\infty} A_j\right)$ and therefore $Z = \bigcup_{i=1}^{\infty}\left(Z \cap \left(K_1 - \bigcup_{j=1}^{\infty} A_j\right)\right)$. Furthermore, for every natural number i we have $Z \cap \left(K_i - \bigcup_{j=1}^{\infty} A_j\right) = Z \cap K_i \subseteq (K_i - A_i)$, therefore Z is the denumerable union of sets of the first Bairecategory and therefore Z is of the first Baire category as desired.

Remark 1. The statement of Theorem 5 does not remain true if in it "denumerable" is replaced by "nondenumerable". If $N = \{a, b, c, \dots\}$ is a set that lacks the Baire property ([**13**], pg. 24) then every singleton $\{a\}$, $\{b\}$, $\{c\}, \dots$ is an outer Baire kernel of itself, however the union of these singletons is N which lacks the Baire property and hence cannot be the outer Baire kernel of any set.

Also. examples can be given to show that the statement of Theorem 5 does not remain true if in it "denumerable" is replaced by the first nondenumerable cardilan $\aleph_1$ (even if we assume $\aleph_1 < 2^{\aleph_0}$). This is because there are models for ZF in which $\aleph_1 < 2^{\aleph_0}$ in which there are non-Baire sets of cardinality $\aleph_1$ [**6**].

However in spite of the remark in the last paragraph we have the following theorem.

Theorem 6. *Let $\aleph$ be any cardinal satisfying $\aleph < 2^{\aleph_0}$. Then it is consistent (with usual axioms of ZF) to assume that the union of the outer Baire kernels of $\aleph$ manuy sets is an outer Baire kernel of the union of these sets.*

Proof. It is knowm ([**2**], pg. 114) that Martin's axiom is consistent with the usual axioms of ZF. Furthermore it is known ([**5**], pg. 286–288) that Martin's axiom implies that if $\aleph < 2^{\aleph_0}$ then (i) the union of $\aleph$ many Baire sets is a Baire set and (ii) the union of $\aleph$ many sets of the first Baire category is a set of the first Baire category. Therefore, using (i) and (ii), a proof following the lines of the proof of Theorem 5 goes through.

The following results is an analogue of Theorem 7 of Abian.

Theorem 7. *Let $\aleph$ be a cardinal. Let us assume that the union of any collection of $\aleph$ many sets of the first Baire category is always a set of the first Baire categor. Then the intersection (as well as the nion) of any collection of $\aleph$ many Baire sets is a Baire set.*

Proof. The proof of this theorem is an exact analogue of the proof of Theorem 7 in Abian's work, but is included for completeness. Let $(B_i)_{i\in\aleph}$ be a collection of $\aleph$ many Baire sets M_i. Let K be an outer Baire kernel of $\bigcap\limits_{i\in\aleph} B_i$. Clearly,

(*) $K - \bigcup\limits_{i\in\aleph}(K - B_i) = \bigcap\limits_{i\in\aleph} B_i$. However, for every $i \in \aleph$ we see that $K - B_i$ is a Baire subset of $K - \bigcap\limits_{i\in\aleph} B_i$ and thus $K - B_i$ is of the first Baire category. But then, by the assumotion of the theorem, $\bigcup\limits_{i\in\aleph}(K - B_i)$ is a set of the first Baire category, which by (*) implies that $\bigcap\limits_{i\in\aleph} B_i$ is a Baire set as desired. That $\bigcup\limits_{i\in\aleph} B_i$ is a Baire set follows by complemention and an application of de Morgan's law.

Remark 2. A measurable outer kernel of a set need be an outer Baire kernel of that set and vice versa. To see this suppose A is a measurable subset of the interval $(0,1)$ that is at the same time a Baire set (for example any Borel subset of $(0,1)$). Let S_1 de a measurable set of the first Baire category such that $m(S_1) > 0$ and $S_1 \cap (0,1) = \varnothing$, where m denotes Lebesgue measure. Let S_2 be a measurable set of the second Baire category such that S_2 is a Baire set, $m(S_2) = 0$ and $S_2 \cap (0,1) = \varnothing$. Sets S_1 and S_2 with the precribed properties given above exist (see [**13**]). Then $A \cup S_1$ is an outer Baire kernel of A but is not a measurable outer kernel of A. Also, $A \cup S_2$ is a measurable outer kernel of A but is not an outer Baire kernel of A. Following Abian (his Remark 2) we have the following.

Remark 3. The concept of an inner Baire kernel Q of a set A can be introduced at the dual of an outer Baire kernel by defining Q to be a Baire subset of A such that the only Baire subset of $A - Q$ are sets of the first Baire category. Also, as expeced the duals of Theorems 1 thrue 6 can then be stated and proved.

Remark 4. If A is a set and $x \in R$, then x is said to be of the second Baire category with respect to A in case neighborhod of x intersets A in a set of the second Baire category. Prof. M. Marjanović has pointed out to the auther that the set $A_2 \cup A$ is an outer Baire kernel of A, where A_2 denotes the set of all points of the second Baire category with respect to A. It is interesting to note that $A_2' \cup A$ is a measurable outer kernel of A, where A_2' denotes the set of all points haing exterior metric density one with respect to A (see [**3**], pg. 180).

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REFERENCES

[1] Abian, A., *Measurable outer kernels of sets*, Publications de l'Institut mathématique, Belgrade, to appear.

[2] Devlin, K. J., *Martin's Axiom*, Logic Colloquium 1977, North–Holand Publishing Co., Amsterdam, 1978.

[3] Goffman, C., *Real Functions*, Holt, Rinehart and Winston, new York, 1953.

[4] Hausdorff, F., *Grundzüge der Mengenlehre*, Leipzig, 1914.

[5] Lévy, *Basic Set Theory*, Springer-Verlag, New York, 1979.

[6] Martin, D. A., Solovay, R. M., *Internal Cohen extensions*, Annals. Math. Logic, **2** (1970).

[7] Miller, H. I., *Relationships between various gauges of the size of real numbers I*, Glasnik Matematički, **9** (29) (1974), 59–64.

[8] Miller, H. I., *Relationships between various gauges of the size of sets of real numbers II*, Akad. Nauka i Umjet. Bosne i Hrcegov. Rad. Odelj. Prirod. Mat. Nauka, LIX (1976), 37–48.

[9] Miller, H. I., *On transformations of Baire sets in* E^N, Akad. Nauka i Umjet. Bosne i hercegov. rad. Odelj. prirod. Mat. Nauka LXI (1978), 75–83.

[10] Miller, H. I., *Some results connected with a problem of Erdoš II*, Proc. Amer. Math. Soc., Vol **75**, No. **2**. 265–268.

[11] Miller, H. I., *An analogue of a theorem of Carathéodory*, Čas. pěst. mat., **106**, 1981, 38–41.

[12] Miller, HI., and Xenikakis, P. J., *Some problems connected with a problem of Erdös I*, Akad. Nauka i Umjet. Bosne i Hercegov. Rad. Odelj. Prior. Mat. Nauka, LXVI (1980), 71–75.

[13] Oxtoby, J. C., *Measure and Category*, Springer-verlag, New York, 1971.

[14] Tall, F. D., *An Alternative to the Continuum Hypothesis and its Uses in General Topology*, University of Toronto, Lecture Notes, 1973.

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