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SOME EMBEDDING THEOREMS

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Summary. In this paper we prove Theorems 1, 2, 3 which are generalization of Malcev theorem ([**6**], p. 274). We also make the correction of a mistake made in our paper [**9**].

1. There are several general theorems on embedding like the following:

Loś theorem ([**4**])

(L) *Let $L_1, L_2 (L_1 \subseteq L_2)$ be languages*[1], *$\mathcal{M}_1$ a model of L_1 and F_2 a set of formulae in L_2. Then $\mathcal{M}_1$ can be extended to some model of F_2 iff for every universal formulae φ in L_1 the following implication holds:*

$$F_2 \vdash \varphi \to \mathcal{M}_1 \models \varphi.$$

The theorem close to the preceding one:

(S) *Let $L_1, L_2 (L_1 \subseteq L_2)$ be languages F_1 and F_2 sets formulae in L_1, L_2 respectively. Then every model $\mathcal{M}_1$ of F_1 can be extended to some model $\mathcal{M}_2$ of F_2 iff for every universal formula φ in L_1 the following implication holds:*

$$F_2 \vdash \varphi \to F_1 \vdash \varphi.$$

Malcev theorem (slighty reformulated):

(M) *Let $L_1, L_2 (L_1 \subset L_2)$ be languages, F_1, F_2 sets of quasiidentities in L_1, L_2 respecively. Then every model $\mathcal{M}_1$ of F_1 can be extended to some model $\mathcal{M}_2$ of F_2 iff for every quasiidentity in L_1 the following implication holds:*

$$F_2 \vdash \varphi \to F_1 \vdash \varphi.$$

Keisler theorem [**5**]

(K) *Let $\mathcal{M}$ be a structure for L, T a theory with language L, Γ a regular set of formulae in L. Then $\mathcal{M}$ has a Γ-extension which is a model of T iff every theorem of T which is disjunction of negations of formulae in Γ is valid in $\mathcal{M}$.*

[1] All languages are of the first order.

Cohn-Rebane theorem ([**1, 11**])

(C R) *Any Ω-algebra can be enbedded in some semigroup.*

A number of results of different authors ([**2, 7, 8, 9, 10, 11**]).

In Theorems (L)—(K) it is supposed $L_1 \subseteq L_2$ which is not the case[2] in (C R) and in some results in [**7, 10, 11**].

We emphasize that Theorem (C R) can be naturally reformulated so that the condition $L_1 \subseteq L_2$ holds. For example, by (C R) the grupoid $\mathcal{G} = (G, \circ)$ where:

$$G = \{a, b\} \qquad \begin{array}{c|cc} \circ & a & b \\ \hline a & b & a \\ b & a & a \end{array}$$

can be embedded in some semigroup $\mathcal{S} = (S, *)$.

According to the proof of (C R) this means that the set G can be extended to some set S, in S can be chosen an element, say $c(c \notin G)$, and can be defined an associative operation $*$ in S such that for all $x, y \in G$ the equality

$$x \circ y = (c * x) * y \tag{1}$$

holds.

As we can see the equality (1)—definition of the operation $\circ$ of the given structure $\mathcal{G}$ by $*$ and the constant symbol c, is required *only* for x, $y \in G$. However, if we permit x, y to be any elements of S, this equality becomes a definition of exactly one operation of S. In such a way in connection with the considered example one model $\mathcal{S}' = (S, \circ, *, c)$ of the language $\{*, \circ, c\}$, which is an expansion of $\{*\}$, appears. In fact, $\mathcal{G}$ is extebded to $\mathcal{S}'$. Similar holds generally in case of embedding a model of one language in some model of some other language providing that in addition certain explicit definitions of operations and relations, like (1), are required.

2. Let $L_1, L_2 (L_1 \subseteq L_2)$ be languages and F_1, F_2 sets of universal Horn formula in L_1, L_2 respectively. Then we have the following theorem.

THEOREM 1. *Every model $\mathcal{M}_1$ of the set F_1 can be extended to some model $\mathcal{M}_2$ of the set F_2 iff for every universal Horn formulae φ in L_1 the following implication holds:*

$$F_2 \vdash \varphi \to F_1 \vdash \varphi. \tag{2}$$

Proof. Only if—part. Let φ be a universal Horn formulae and suppose $F_2 \vdash \varphi$. Further, let $\mathcal{M}_1$ be any model[3] of F_1. Denote by $\mathcal{M}_2$ a model of F_2 which is an extension of $\mathcal{M}_1$ and which exists by hypothesis. Then:

$$\mathcal{M}_2 \models \varphi.$$

[2] In (C R) L_1 is Ω and $L_2 = \{*\}$, where $*$ is a binary operation symbol.

[3] If F_1 is incosistent then we have $F_1 \vdash \varphi$ and the proof is completed.

As $\mathcal{M}_2$ is an extension of $\mathcal{M}_1$ and φ is universal we conclude

$$\mathcal{M}_1 \models \varphi$$

wherefrom by the completeness theorem it follows

$$F_1 \vdash \varphi$$

for φ is true in every model $\mathcal{M}_1$ of F_1.

If part. Let $\mathcal{M}_1$ be a model of F_1. Consired the set

$$(Diag\mathcal{M}_1) \cup F_2 \tag{3}$$

of some formulae in the language $L_2 \cup M$, where *Diag* $\mathcal{M}_1$, in fact equivalent to *the diagram* of $\mathcal{M}_1$, is the set of all formulae having one of the form

(i) $$O(a_1, \ldots, a_m) = a$$

(ii) $$R(b_1, \ldots, b_n)$$

(iii) $$\neg R(c_1, \ldots, c_n)$$

(iv) $$d \neq e$$

which are true in $\mathcal{M}_1$, where $O,\ R \in L_1$, a_i, b_j, c_k, a, d, $e \in M$.

The proof will be completed if we prove that the set (3) is consistent, for then any model $\mathcal{M}_2$ of (3) will be an extension of $\mathcal{M}_1$ and a model of F_2, too. Assume on the contrary that (3) is inconsistent. Then using one general logical fact ([**12**], p. 42) it follows

$$F_2 \vdash \neg(A_1 \wedge \cdots \wedge A_k) \tag{4}$$

for some finitely many elements $A_1, \ldots, A_k$ of *Diag* $\mathcal{M}_1$. Let P be conjuction of all A_i which are of the forms (i), (ii) and let $\neg Q_1, \ldots, Q_r$ be new denotations for the rest of the formulae A_i. Then (4) becomes

$$F_2 \vdash \neg(P \wedge (\neg Q_1 \wedge \cdots \wedge \neg Q_r))$$

which is equivalent to

$$F_2 \vdash \Rightarrow (Q_1 \vee \cdots \vee Q_r). \tag{5}$$

Denote by $s_1, \ldots, s_p$ all elements of M_1 occuring in $P \Rightarrow Q_1(\vee \cdots \vee Q_r)$ and consired the set[4] $F_2 \cup \{P(s_1, \ldots, s_p)\}$. For this set there are two possibilities:

1° *It is inconsistent,* 2° *It is consistent*

Case 1°. Then we have

$$F_2 \vdash \neg P(s_1, \ldots, s_p). \tag{6}$$

[4] P is denoted by $P(s_1, \ldots, s_p)$.

As the constant symbols $s_1, \dots, s_p$ do not appear in F_2, using the related general logical fact ([**12**], p. 33), (8) yields

$$F_2 \vdash (\forall x_1, \dots, x_p) \neg P(x_1, \dots, x_p) \tag{7}$$

where the variables $x_1, \dots, x_p$ do not appear in F_2. The formula

$$(\forall x_1, \dots, x_p) \neg P(x_1, \dots, x_p)$$

is in the language L_1 and obviously is equivalent to a universal Horn formula. Using the hypothesis (2) from (7) we obtain

$$F_1 \vdash (\forall x_1, \dots, x_p) \neg P(x_1, \dots, x_p)$$

wherefrom

$$\mathcal{M}_1 \models (\forall x_1, \dots, x_p) \neg P(x_1, \dots, x_p)$$

and particularly

$$\mathcal{M}_1 \models \neg P(s_1, \dots, s_p)$$

which contradicts $\mathcal{M}_1 \models P(s_1, \dots, s_p)$. Thus it is not possible that

$$F_2 \cup \{P(s_1, \dots, s_p)\}$$

is inconsistent.

Case 2°. If $F_2 \cup \{P(s_1, \dots, s_p)\}$ is consistent then it has a model. Denote by $\mathcal{F}(s_1, \dots, s_p)$ the free model of this set generated by all constant symbols occuring in it[5]. From (5) we deduce

$$F_2,\, P \vdash Q_1 \vee \cdots \vee Q_r$$

wherefrom it follows

$$\mathcal{F}(s_1, \dots, s_p) \models Q_1 \vee \cdots \vee Q_r$$

which implies that at least one of the formulae $Q_1, \dots, Q_r$, Q_i say, is true in $\mathcal{F}(s_1, \dots, s_p)$. Thus

$$\mathcal{F}(s_1, \dots, s_p) \models Q_i.$$

From this and the definition of freee model it follows

$$F_2,\, P(s_1, \dots, s_p) \vdash Q_i(s_1, \dots, s_p)$$

wherefrom by Deduction theorem:

$$F_2 \vdash P(s_1, \dots, s_p) \;\Rightarrow\; Q_i(s_1, \dots, s_p). \tag{8}$$

[5]Its elements are equivalence classes of the set Term $(L_2, s_1, \dots, s_p)$—the set of all variable-free terms in the language $L_2 \cup \{s_1, \dots, s_p\}$, with respect to the relation $\sim$ defined by

$$t_1 \sim t_2 \quad \text{iff} \quad F_2, P \vdash t_1 = t_2.$$

Operations and relations with equivalence classes are defined in the usual way (see Definition in the part 3.)

As $s_1, \dots, s_p$ do not appear[6] in L_2, (8) yields

$$F_2 \vdash (\forall x_1, \dots, x_p)\,(P(x_1, \dots, x_p) \Rightarrow Q_i(x_1, \dots, x_p)) \tag{9}$$

where the variables $x_1, \dots, x_p$ do not appear in F_2.

Using the hypothesis (2) from (9) we deduce

$$F_1 \vdash (\forall x_1, \dots, x_p)\,(P(x_1, \dots, x_p) \Rightarrow Q_i(x_1, \dots, x_p))$$

wherefrom it follows

$$\mathcal{M}_1 \models (\forall x_1, \dots, x_p)\,(P(x_1, \dots, x_p) \Rightarrow Q_i(x_1, \dots, x_p))$$

and particulary

$$\mathcal{M}_1 \models P(s_1, \dots, s_p) \Rightarrow Q_i(s_1, \dots, s_p)$$

which contradicts the assumptions

$$\mathcal{M}_1 \models P(s_1, \dots, s_p)\, \mathcal{M}_1 \models \neg Q_i(s_1, \dots, s_p).$$

The proof of the theorem is completed.

3. Analyzing the proof of Theorem 1 the following facts can be noticed:

(j) If F_1, F_2 are quasiidentities Case 1° in fact does not appear and consequnetly the formulae φ may be supposed to be quasiidentities. In such a way Theorem 1 yields Theorem (M).

(jj) The assumption that F_1 is a set of universal Horn formulae in fact has not been employed and consequently we have the following theorem.

THEOREM 2. *Let L_1, $L_2 (L_1 \subseteq L_2)$ be languages, F_1, F_2 sets of formulae in L_1, L_2 respectively such that the elements of F_1 are arbitrary and elements of F_2 are universal Horn formulae. Then every model $\mathcal{M}_1$ of F_1 can be extended to some model $\mathcal{M}_2$ of F_2 iff for every universal Horn formula φ in L_1 the implication*

$$F_2 \vdash \varphi \rightarrow F_1 \vdash \varphi$$

holds.

(jjj) The crucial point in the proof of Theorem 1 is the pass from (5) to either (6) or (8), i.e. the pass from

$$F_2 \vdash P \Rightarrow (Q_1 \vee \cdots \vee Q_r)$$

to

$$\text{either } F_2 \vdash \neg P \text{ or } F_2 \vdash P \Rightarrow Q_i, \text{ for some i}$$

which is grounded on the fact that the set $F_2 \cup \{P\}$ has a free model. Bearing this in mind a new generalization can be formulated, in which we replace the word *free* with *deductive*.

[6] As a mater of fact in the very begining it is supposed $L_2 \cap M_1 = \varnothing$.

Theorem 3. *Let F_1 be any set of formulae in the language L_1 and F_2 a set formulae in the language $L_1 (L_1 \subseteq L_2)$ having the property:*

The set $F_2 \cup Q$ where Q is any finite set of formulae of the form

$$O(a_1, \dots, a_m) = a, \quad R(b_1, \dots, b_n)$$

($O,\, R \in L_2,\, |O| = m,\ |R| = n$ are operation and relation symbols and a, a_i, b_j are constant symbols not in L_2) has a deductive model in the sense of Definition[7] *1.*

Then every model $\mathcal{M}_1$ of F_1 can be extended to some model $\mathcal{M}_2$ of F_2 iff for every universal Horn Formula φ in L_1 the implication

$$F_2 \vdash \varphi \to F_1 \vdash \varphi$$

holds.

(jw) Theorem (S) has the similar proof. Namely, it sufficies to keep the part of the preceding proof untill the step (5) and to continue in the following way:

Denoting the formula

$$P \;\Rightarrow\; (Q_1 \vee \cdots \vee Q_r)$$

by $\varphi(s_1, \dots, s_n)$ (5) becomes

$$F_2 \vdash \varphi(s_1, \dots, s_p)$$

which yields

$$F_2 \vdash (\forall x_1, \dots, x_p)\varphi(x_1, \dots, x_p) \quad (x_i \text{ are variables, not in } F_2)$$

and by hypothesis (2)

$$F_1 \vdash (\forall x_1, \dots, x_p)\varphi(x_1, \dots, x_p). \tag{10}$$

As $\mathcal{M}_1$ is a model for F_2 (10) juields

$$\mathcal{M}_1 \models (\forall x_1, \dots, x_p)\varphi(x_1, \dots, x_p)$$

and therefore particularly

$$\mathcal{M}_1 \models \varphi(s_1, \dots, s_p)$$

which by definition of *Diag* $\mathcal{M}_1$ contradicts to the fact

$$\mathcal{M}_1 \models \neg\varphi(s_1, \dots, s_p).$$

We give now the mentioned definition of deductive model.

Definition 1. Let F be a set of formulae of the language L, $C = \{c_i | i \in I\} \neq \varnothing$ the set of all constant symbols occurring in F and *Term*(L, C) the set of all

[7] See the sequel of this part of the paper.

variable-free terms in L. In the set *Term* (L, C) we define the relation $\sim$ in the following way

$$t_1 \sim t_2 \quad \text{iff} \quad F \vdash t_1 = t_2.$$

It is clear that $\sim$ is an equivalence relation. In the set $D = Term(L, C)/\sim$ for each operation symbol O and relation symbol $R(O,\, R \in L,\ |O| = m,\, |R| = n)$ we define the operation $O/\sim$ and relation $R/\sim$:

$$O/\sim(C_{t_i}, \dots, C_{t_m}) = C_t \quad \text{iff} \quad F \vdash O(t_1, \dots, t_m) = t,$$
$$R/\sim(C_{t_1}, \dots, C_{t_n}) \quad \text{iff} \quad F \vdash R(t_1, \dots, t_n).$$

Generally, in such way we obtain a model $\mathcal{D}$ of the language L which is not necessarily a model for F. If that is the case, we say that $\mathcal{D}$ in the[8] *deductive model* for F and that F *has that deductive model*.

For instance, the following sets have deductive models, providing their consistency:

1° Sets of quasiidentities

2° Sets of universal Horn formulae

3° Henkin complete sets, i.e. which with each formula of the form $(\exists x)\varphi(x)$ (x is the only free variable in φ) have some theorem of the form $(\exists x)\varphi(x) \Rightarrow \varphi(c)$—$c$ is a constant symbol

4° The set of axioms of formal arithmetic (of the first order).

4. In our paper [**9**] in the formulation of Theorem 1 and 2 instead of the condition that the operation $\circ$ satisfies no nontrivial algebaric laws should be the condition:

$\circ$ *satisfies no nontrivial quasiidentities*

which is an accordance with Malcev theorem. The mistake has been noticed by Professor Gorgi Čupona, University of Skoplje to whom we would like to express our greatfulness.

It still remains open the problem:

What conditions for the sets of algebraic laws F_1, F_2 (in the languages $L_1, L_2 (L_1 \subseteq L_2)$ respectively) are necessary and sufficient for the following equivalence:

Every model $\mathcal{M}_1$ of F_1 can be extended to some model M_2 of F_2 iff for any algebraic law φ in L_1 the implication

$$F_2 \vdash \varphi \to F_1 \vdash \varphi$$

holds.

The solution of this problem would also be the solution of the problems which still remain in connection with our paper[**9**].

[8]In [**12**] the term canonical structure is used.

REFERENCES

[1] Cohn P. M., *Universal Algebra*, Harper Row, New York, Evanston, London and John Weatherhill, Inc. Tokyo, 1965, pp. 184–186.

[2] Čupona G., *Za teoremata na Kon-Rebane*, God. Zb. PMF **20**, A (1970), Skopje pp. 5–14.

[3] Čupona G., Markovski S., *Smestuvanje na uneverzalni algebri*, God. Zb. PMF **25-26** (1975) (1976), sek. A, Skopje pp. 15–34.

[4] Loš J., *On the extending of models I*, Fund Math. **42** (1955), pp.38–54.

[5] Keisler J. H., *Theory of models with generalized atomic formulas*, J. Symb. Logic **25** (1960), pp. 1–26.

[6] Мальцев А. И. *Алгебрачиеские системы*, Наука, Москва 1970.

[7] Markovski S., *On quasivarieties of generalized subalgebras*, Algebraic conference, Skopje 1980, pp. 125–129.

[8] Pigozzi D., *Univezal equtional theories and variaties of algebras*, Annals of Math. Logic **17** (1979), pp. 117–150.

[9] Prešić M. D., Prešić S. B., *On the embedding of Ω-algebras in grupoids*, Publ. Inst. Math. Beograd, **21** (**35**), 1977, pp. 169–174.

[10] Radojčić M. D. (now Prešić M.D.), *On the embedding of universal algebras in grupoids holding the law* $xy * zu ** = xz * yu **$, Mat. vesnik, Beograd, **5** (1968), pp. 353–356.

[11] Ребяне О. К., *О представлении универсальных алгебр в коммутативных подгрупах*, Сибир. мат. ж. **7** (1966), 878–883.

[12] Shoenfield, J. R., *Mathematical Logic*, Addison-Wesley, Reading-Maassachusetts, Menlo Park-California, London, Don Mills-Ontario, 1967.

Correction

Marica D. Prešić, A convergence theorem for a method for simultaneous determination of all zeros of a polynomial, Publ. Inst. Math., Beograd, **28** (42), 1980, *pp.* 159–168.

Throughout the paper instead of the symbol σ it should be written 6 (the number six). Apart from this in Abstract the letter s in *Ostrowski's* is omitted. Further, on the page 161, the second line from the bottom instead of $a_i, \dots s_i$ it should be $\{a_i, \dots, s_i\}$ and in Acknowledgement, on the page 165 instead of *mode* it should be *made*.

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