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ONE OF THE POSSIBLE FORMAL DESCRIPTIONS OF DEDUCIBILITY

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Abstract. Having in mind different investigations of implication, i. e., of the logical consequence relation, we will try to point out a general kernel of formal systems in which the deducibility relation is stated in the system itself. In connection with any formal theory θ we observe a formal theory $\theta(\rightarrow)$ which is able to define the fundamental factor of *θ-deducibility.* By showing that the basic binary relation of $\theta(\rightarrow)$ is just a formal description of the metatheoretic deducibility relation of θ, the essential statement, the assertion 2.9., justifies contemplation of a formal theory like $\theta(\rightarrow)$. Furthermore, by the assertions 3.3 and 3.4. an interesting conection between formal theories $\theta(\sim)$ (cf. [**1**]) and $\theta(\rightarrow)$ is given.

1. Gentzen's idea of sequent calculi and the paper [**1**] of S. B. Prešić have influenced immediately on this contemplation. We will try to describe a procedure by which we can assign an *inequational formal theory*, i. e., a formal theory in which some binary predicate is a preordering, to any formal theory. As thingc stand, a deduction relation (denoted by $\vdash$ and defined in the usual way) is a preordering, relation and so every logical system is in connection with some preordered systems.

2. Let θ be a formal theory. By $\theta(\rightarrow)$ we will denote a formal theory defined as follows:

2.1. The set of basic symbols of the initial formal theory θ will be extended by three **new** symbols: $\top$, & and $\rightarrow$; $\top$ is a new constant, & is a binary operational symbol and $\rightarrow$ is a binary predicate symbol.

The fundamental notions are defined as usual:

2.2. *Definition.* (i) $\top$ and all formulas of θ are *preformulas* of $\theta(\rightarrow)$.

(ii) If A and B are *preformulas* of $\theta(\rightarrow)$, then $\&AB$ is a *preformula* of $\theta(\rightarrow)$.

(iii) *Preformulas* are only those expressions obtained by (i) and (ii)

2.3. *Definition.* If A and B are preformulas of $\theta(\rightarrow)$, then $A \rightarrow B$ is a *formula* of $\theta(\rightarrow)$.

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The set of all formulas of the formal theory θ will be denoted by For(θ). Henceforth, A, B, C, D, $A_1, \ldots$ and $F, G, F_1, G_1, \ldots$ will be metavariables ranging over preformulas of $\theta(\to)$ and formulas of θ, respectively.

2.4. *Axioms* of $\theta(\to)$ are defined by the following:

(i) $F \to F$, $F \to \top$, $\& AB \to \& BA$.

(ii) If F is an axiom (schemes) of θ, then $\top \to F$ is an axiom (scheme) of $\theta(\to)$.

(iii) If $\frac{G_1, \ldots, G_{k-1}, G_k}{G}$ is a rule (scheme) of inference of θ, then the formula $\& G_1 \& \ldots \& G_{k-1}\, G_k \to G$ is an axiom (scheme) of $\theta(\to)$.

(iv) Axioms are only those formulas obtained by (i), (ii) and (iii),

2.5. The *rule schemes* of $\theta(\to)$ are

$$(\mathrm{IKS})\frac{A \to B \quad A \to C}{A \to \& BC}, \quad (\mathrm{IKA})\frac{A \to B}{\& AC \to B}, \quad (\mathrm{TR})\frac{A \to B \quad B \to C}{A \to C}.$$

2.6. Lemma. (1) $\vdash_{\theta(\to)} A \to A$;
(2) $\vdash_{\theta(\to)} \&\& ABC \to \& A \& BC$; $\vdash_{\theta(\to)} \& A \& BC \to \&\& ABC$;
(3) $\vdash_{\theta(\to)} A \to \top$; $\vdash_{\theta(\to)} \& A\top \to A$;
(4) *if* $\vdash_{\theta(\to)} A \to B$ *and* $\vdash_{\theta(\to)} C \to D$ *then* $\vdash_{\theta(\to)} \& AC \to \& BD$;
(5) $\vdash_{\theta(\to)} \& AB \to C$ *iff* $\vdash_{\theta(\to)} \& BA \to C$;
(6) $\vdash_{\theta(\to)} A \to \& BC$ *iff* $\vdash_{\theta(\to)} A \to \& CB$;
(7) $\vdash_{\theta(\to)} \&\& ABC \to D$ *iff* $\vdash_{\theta(\to)} \& A \& BC \to D$;
(8) $\vdash_{\theta(\to)} A \to \&\& BCD$ *iff* $\vdash_{\theta(\to)} A \to \& B \& CD$;
(9) *if* $A \to \& BC$ *is* k*-provable in* $\theta(\to)$, *then* $A \to B$ *is* k_1*-provable and* $A \to C$ *is* k_2*-provable in* $\theta(\to)$, *where* $k_1, k_2 < k$[1].

Proof. (1)—(8) Directly—using the rule schemes of $\theta(\to)$. (9) By induction on k.

According to 2.6. Lemma (5)—(8), we see that the preformulas $\& AB$ and $\&\& ABC$ can be replaced by $\& BA$ and $\& A \& BC$, respectively, in the framework of $\theta(\to)$, and conversely[2].

2.7. Lemma. *Iff* $\vdash_{\theta} F$, *then* $\vdash_{\theta(\to)} \top \to F$.

Proof. By induction on the length of the proof for F in θ.

2.8. Corollary. *If* $\vdash_{\theta} F_1$ *and* $\ldots$ *and* $\vdash_{\theta} F_n$, *then* $\vdash_{\theta(\to)} \top \to F_1 \& \ldots \& F_n$ $(n \geq 1)$.

2.9. Theorem. $F_1 \ldots, F_n \vdash_{\theta} F$ *iff* $\vdash_{\theta(\to)} F_1 \& \ldots \& F_n \to F$ $(n \geq 1)$.

[1] F is k-provable in θ iff the length of the shortest proof of F in θ is k.
[2] So, we can write $A \& B$ instead of $\& AB$.

Proof. The "only if" part. By induction on the length m of the proof of $F_1, \dots, F_n \vdash_\theta F$.

Case $m = 1$: If F is the axiom of θ, then $\top \to F$ is the axiom of $\theta(\to)$. By 2.6. Lemma (3) $\vdash_{\theta(\to)} F_1 \& \dots \& F_n \to \top$, and by (TR) we have $\vdash_{\theta(\to)} F_1 \& \dots \& F_n \to F$. If F is F_1 for some $1 \le i \le n$, say that F is F_1, then $F_1 \to F$ is the axiom of $\theta(\to)$. Hence, using the rule scheme (IKA) we have $\vdash_{\theta(\to)} F_1 \& \dots \& F_n \to F$.

Case $m > 1$: The following subcases are possible:

(i) F is either the axiom of θ or F_1 (for some $1 \le i \le n$);

(ii) F is the consequence of some preceding formulas by the rule (scheme) $\frac{G_1, \dots, G_k}{F}$. The subcase (i) is the same as the case $m = 1$. Subcase (ii): by induction hypothesis: $\vdash_{\theta(\to)} F_1 \& \dots \& F_n \to G_1$,

$\dots\dots\dots\dots\dots\dots\dots$

$\vdash_{\theta(\to)} F_1 \& \dots \& F_n \to G_k$;

using rule scheme (IKS) (K-1) times, we have

$$\vdash_{\theta(\to)} F_1 \& \dots \& F_n \to G_1 \& \dots \& G_k.$$

$G_1 \& \dots \& G_k \to F$ is the axiom scheme of $\theta(\to)$. Therefrom, by (TR) we derive $\vdash_{\theta(\to)} F_1 \& \dots \& F_n \to F$.

The "if" part. By induction on the number m of uses of the rule schemes (IKS), (IKA) and (TR).

Case $m = 0$: $F_1 \& \dots \& F_n \to F$ is the axiom of $\theta(\to)$. Then either $\frac{F_1, \dots, F_n}{F}$ is the rule of θ or $n = 1$ and F_1 is F or F_1 is $\top$ or F is $\top$. In any case we have $F_1, \dots, F_n \vdash_\theta F$.

Case $m > 0$: The following subcases are possible:

(i) in the last step of the proof we used the rule scheme (IKA) on the formula $F_1 \& \dots \& F_i \to F$ (for some $1 \le i \le n$);

(ii) in the last step of the proof we used the rule scheme (TR) on the formulas $F_1 \& \dots \& F_n \to G_1 \& \dots \& G_k$ and $G_1 \& \dots \& G_k \to F$. Subcase (i): by induction hypothesis

$$F_1, \dots, F_i \vdash_\theta F$$

and so $F_1, \dots, F_i,\ F_{i+1}, \dots, F_n \vdash_\theta F$. Subcase (ii): by 2.6 Lemma (9) we have

$\vdash_{\theta(\to)} F_1 \& \dots \& F_n \to G_1$,

$\dots$

$\vdash_{\theta(\to)} F_1 \& \dots \& F_n \to G_k$, and by induction hypothesis:

$F_1, \dots, F_n \vdash_\theta G_1$,

$\dots$

$F_1, \dots, F_n \vdash_\theta G_k$,

therefrom $F_1, \dots, F_n \vdash_\theta F$ which was to be demonstrated.

In accordance with 2.7. Lemma, and 2.9. Theorem, we see that the binary predicate $\to$ of $\theta(\to)$ is, in fact, a formalization of the deduction relation $\vdash_{\theta}$, the binary operation & is related to the metatheoretic "and", while the constant $\top$ characterizes the set $\mathrm{Th}(\theta)$ of all theorems of θ in the sense that $\mathrm{Th}(\theta) = \{F | \top \to F$ is provable in $\theta(\to)\}$.

Assording to (TR) and 2.6. Lemma (1), we will call $\theta(\to)$ an *inequational* description of θ.

2.10. THEOREM. $\mathrm{For}\,(\theta) = \mathrm{Th}\,(\theta)$ *iff* $\mathrm{For}\,(\theta(\to)) = \mathrm{Th}\,(\theta(\to))$.

Proof. Let $F_1\& \dots \& F_n \to G_1\& \dots \& G_m$ be any formula of $\theta(\to)$. If $\mathrm{For}(\theta) = \mathrm{Th}(\theta)$, then, in accordance with 2.7. Lemma, $\vdash_{\theta(\to)} \top \to G_1, \dots, \vdash_{\theta(\to)} \top \to G_m$; so using the rule scheme (IKS) $\vdash_{\theta(\to)} \top \to G_1\& \dots \& G_m$. According to 2.6. Lemma (3), $\vdash_{\theta(\to)} F_1\& \dots \& F_n \to \top$, and so, by (TR) we have $\vdash_{\theta(\to)} F_1\& \dots \& F_n \to G_1\& \dots \& G_m$. On the other hand, if $\mathrm{For}\ (\theta(\to)) = \mathrm{Th}(\theta\ \to))$, then all formulas of the form $\top \to F$ are theorems of $\theta(\to)$, and according to 2.9. Theorem $\mathrm{Th}(\theta)=\mathrm{For}(\theta)$.

Consequently, (syntactical) consistency is preserved in passing from θ to $\theta(\to)$.

3. Let $A \leftrightarrow B$ iff $A \to B$ and $B \to A$ in $\theta(\to)$.

3.1. COROLLARY. $\vdash_{\theta} F$ *iff* $\vdash_{\theta(\to)} F \leftrightarrow \top$.

3.2. LEMMA. (1) $\vdash_{\theta(\to)} A \leftrightarrow A$; (2) $\vdash_{\theta(\to)} A\&\top \leftrightarrow A$; $\vdash_{\theta(\to)} A\&A \leftrightarrow A$; (3) *if* $A \leftrightarrow B$, *then* $B \leftrightarrow A$; (4) *if* $A \leftrightarrow B$ *and* $B \leftrightarrow C$, *then* $A \leftrightarrow C$; (5) *if* $A \leftrightarrow B$ *and* $C \leftrightarrow D$, *then* $A\&C \leftrightarrow B\&D$; (6) *if* $A \to B$, *then* $A \leftrightarrow A\&B$.

Proof. For instance (6). Of course, $A \to B$ and $A \to A$, by rule scheme (IKS) we derive $A \to A\&B$. On the other hand, from $A \to A$, by rule scheme (IKA) we derive $A\&B \to A$.

3.3. Let $\theta(\sim)$ be the equational formal theory described in [**1**] of S. B. Prešić. The following statement is the consequence of the preceding Lemma.

3.4 LEMMA. *If* $\vdash_{\theta(\sim)} A \sim B$, *then* $\vdash_{\theta(\to)} A \leftrightarrow B$.

3.5. THEOREM. *If* $\vdash_{\theta(\to)} A \leftrightarrow B$, *then* $\vdash_{\theta(\sim)} A \sim B$.

Proof. Let $A \leq B$ in $\theta(\sim)$ iff $A\&B \sim A$ in $\theta(\sim)$. First we will prove that if $A \to B$ in $\theta(\to)$, then $A \leq B$ in $\theta(\sim)$. From $F\&F \sim F$ we have $F \leq F$; if F is an axiom of θ, then $\top\&F \sim \top$ in $\theta(\sim)$, and $\top \leq F$; if $A\&\top \sim A$ in $\theta(\sim)$, then $A \leq \top$ in $\theta(\sim)$; for a rule of inference $\frac{G_1, \dots, G_k}{G}$ of θ, $G_1\& \dots \& G_k\&G \sim G_1\& \dots \& G_k$ is an axiom of $\theta(\sim)$, therefrom $G_1\& \dots \& G_k \leq G$; if $A \leq B$ and $A \leq C$, then $A \leq B\&C$; if $A \leq B$, then $A\&C \leq B$; if $A \leq B$ and $B \leq C$, i.e. $A\&B \sim A$ and $B\&C \sim B$, then we derive immediately $A\&B\&C \sim A\&C$ and $A\&B\&C \sim A\&B$, and so $A\&C \sim A$, i.e. $A \leq C$. Also, $A \leq B$ and $B \leq A$ iff $A \sim B$, Therefore, if $A \to B$ and $B \to A$ in $\theta(\to)$, then $A \leq B$ and $B \leq A$ in $\theta(\sim)$, i.e. $A \sim B$ in $\theta(\sim)$.

4. THEOREM. *If θ is a formal theory in a language containing the binary operations $\wedge, \Rightarrow$ such that* (1) $F, G \underset{\theta}{\vdash} F \wedge G$, $F \wedge G \underset{\theta}{\vdash} F$ *and* $F \wedge G \underset{\theta}{\vdash} G$; *and* (2) $F \underset{\theta}{\vdash} G$ *iff* $\underset{\theta}{\vdash} F \Rightarrow G$[3], *then* $\underset{\theta(\rightarrow)}{\vdash} F_1 \& \ldots \& F_n \rightarrow G_1 \& \ldots \& G_m$ *iff* $\underset{\theta}{\vdash} F_1 \wedge \ldots \wedge F_n \Rightarrow G_1 \wedge \ldots \wedge G_m (m, n \geq 1)$.

Proof. This can be proved almost in the same way as Theorem 3 in [**1**].

In this case, when conditions (1) and (2) of the preceding theorem are satisfied, by mapping g: For $(\theta(\rightarrow)) \rightarrow \mathrm{For}(\theta)$, defined by equality $g(F_1 \& \ldots \& F_n \rightarrow G_1 \& \ldots \& G_m) = F_1 \wedge \ldots \wedge F_n \Rightarrow G_1 \wedge \ldots \wedge G_m$[4], the theorems of $\theta(\rightarrow)$ will be mapped into theorems of θ. We will call the formal theory $\theta(\rightarrow)$ an *inequational reformulation* of θ.

4.1. It can be proved (cf. [**1**] and [**2**]) that the "corresponding" formal theories $\theta(\sim)$, and consequently $\theta(\rightarrow)$, are in the cases of the intuitionistic propositional calculus, classical propositional calculus and classical first-order predicate calculus (these are the cases of the equational (inequational) reformulations) just corresponding algebras: pseudo-Boolean. Boolean and cylindric. Furthermore, there is an isomorphism between $\theta(\sim)$ (or $\theta(\rightarrow)$) and the Lindenbaum algebra of θ.

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[3] Cf. Cond. 1. and Cond. 2. in [**1**].

[4] If $F_i = \top$ or $G_j = \top$ in $\theta(\rightarrow)$ (for some $1 \leq i \leq n$ or $1 \leq j \leq m$), then F_i or G_j in θ must be replaced by some axiom of θ.