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SPECTRAL CHARACTERIZATIONS OF LINE GRAPHS. VARIATIONS ON THE THEME

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Abstract. The following three topics related to spectral characterizations of line graphs are discussed: local structure of regular graphs with the least eigenvalue–2 which contain $K_{1,3}$ as an induced subgraph, switching regular line graphs into regular graphs which are not line graphs, and line switching (of graphs) as a modification of familiar (vertex) switching.

0. Introduction

There is an extensive literature about spectral characterization of line graphs (see, for example, [**6**], Section 6.3). Eigenvalues of line graphs are bounded from below by -2. Conversely, connected graphs G with the least eigenvalue $\lambda(G) = -2$ are generalized line graphs with a finite number of exceptions. There are exactly 187 connected regular graphs with the least eigenvalue -2 which are not generalized line graph (i.e. neither line graphs nor cocktail party graphs [**2**]). These graphs are called exceptional graphs. Exactly 68 exceptional graphs are cospectral with some line graphs. All of them can be obtained by switching line graphs with which they are cospectral.

In this paper we shall discuss some details related to these topics. In Section 1. we shall study the local structure of exceptional graphs having $K_{1,3}$ as an induced subgraph. Section 2 contains a new, quite simple, way to find regular line graphs which can be switched into exceptional graphs. Switching line graphs is interpreted in Section 3. as a line switching of the root graphs and some interesting effects are described.

1. Exceptional graphs containing $K_{1,3}$

Suppose $K_{1,3}$ is an induced subgraph of a regular graph G with $\lambda(G) \geq -2$ (see Fig. 1).

Graphs on Fig. 2 all have least eigenvalue less than -2 and therefore they are forbidden induced subgraphs for G.

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We should add new edges to the graph on Fig. 1 in order to avoid all these forbidden subgraphs.

Vertices 3 and 5 cannot be adjacent since otherwise a forbidden subgraph G_1 induced by vertices 1,2,3,4,5, would appear. Hence, new edges can occur only on level 2. Let H and H' be subgraphs of G induced by vertices $1, 2, \dots, 10$ and $5, 6, \dots, 10$, respectively.

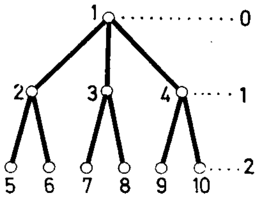

Fig. 1

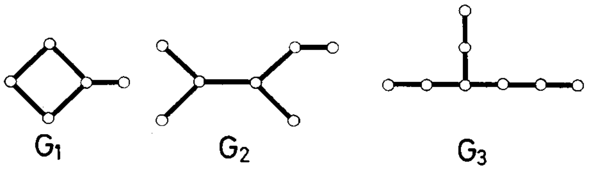

Fig. 2

Interactive programming system "Graph" (see [**7**]) has been used to study the structure of additional edges on level 2. The graphs considered has been modified by a light pen end the eigenvalues repeatedly computed.

H' with less than 6 edges such that $\lambda(H) \geq -2$ has not been found. This led to the idea that H' should have a relatively large number of edges and in particular that H' is regular of degree 2. In that case H would be a cubic graph which would be a nice result. Although cubic structures are constructed below, examples of H' for several exceptional graphs G, easily analysed in the form they are constructed in [**3**], show that H' need not be regular.

All these facts are quite interesting when related to exceptional graphs which are cospectral to line graphs. As shown in [**3**] all the 68 graphs with only one exception contain $K_{1,3}$ as an induced subgraph. However, other exceptional graphs need not contain $K_{1,3}$ as we shall see below.

We are now in a position to construct cubic exceptional graphs, found in [**2**] using a computer produced table of cubic graphs.

A cubic exceptional graph must contain, as an induced subgraph, one of nine Beineke's forbidden subgraphs [**1**]. Since the degree is low only a small number of them can really occur. They are displayed on Fig. 3.

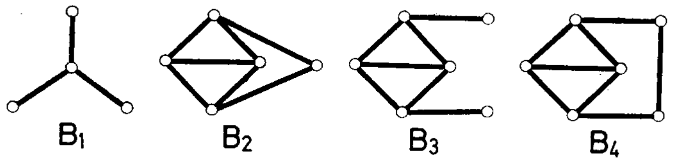

Fig. 3

Let us first find cubic exceptional graphs containing $B_1 = K_{1,3}$.

If edges $5,6; 7,8; 9,10$ exist in Fig. 1 then we forceably get C_1 from Fig. 4.

If the edge $5,6$ does not exist each of vertices $7,8,9,10$ has to be adjacent to 5 or 6 (to avoid G_2). We get C_2, C_3, C_3', C_4 (see Fig. 4). Graphs C_3 and C_3' are isomorphic and C_4 is the Petersen graph. In neither of these cases we can add new vertices since otherwise one of forbidden subgraphs from Fig. 2 would occur.

Let us now turn to the cubic graphs which do not contain $K_{1,3}$. If a cubic graph contains B_2 it contains also B_1 and we have previous case. Suppose G contains B_3 or B_4 and not B_1. Then G must contain one graphs from Fig. 5 as an induced subgraph or G is the graph C_5 from Fig. 4. However graphs from Fig. 5 have the least eigenvalue less than –2. Hence, the only new graph is C_5.

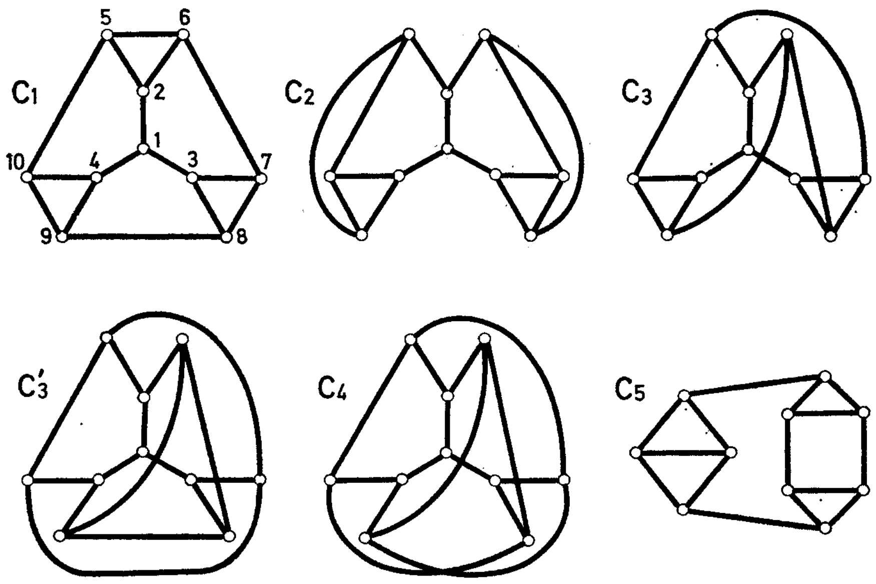

Fig. 4

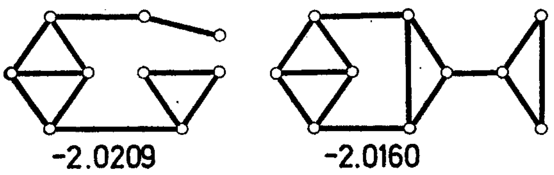

Fig. 5

2. Exceptional graphs switching equivalent to regular line graphs

The following theorem is contained in [**4**]: *If a regular graph of degree r with n vertices can be swiched into a regular graph of degree $r*$, then $r * -n/2$ is an eigenvalue of G.*

Example 2 of the same paper is an application of this theorem to $L(K_8)$. It is proved that if $s > 8$, $L(K_8)$ cannot be switched into another regular graph of the same degree. Hence, exceptional graphs switching equivalent to $L(K_8)$ do not exist if $s > 8$.

This example can be generalized to all regular line graphs.

Let $L(G)$ be regular with G connected.

(1) Suppose first G is regular degree r with n vertices. $L(G)$ is of degree $2r-2$ and has $nr/2$ vertices. If $L(G)$ can be swiched into another regular graph of the same degree then $2r-2-nr/4$ is an eigenvalue of $L(G)$. Obvioulsy, $2r-2-nr/4 \geq -2$, which implies $n \leq 8$.

(2) Let G be semiregular bipartite with parameters $n_1, d_1, n_2, d_2 (n_1 d_1 = n_2 d_2)$. $L(G)$ has $n_1 d_1$ vertices and degree $d_1 + d_2 - 2$. Therefore, we have

$$d_1 + d_2 - 2 - n_1 d_1/2 \geq -2, \quad n_1 d_1 \leq 2(d_1 + d_2),$$
$$n_1 \leq 2(1 + d_2/d_1) = 2(1 + n_1/n_2), \quad 1/n_1 + 1/n_2 \geq 1/2.$$

Let $n_1 \leq n_2$. If $n_1 = 1$, then $L(G) = K_{n_2}$ and there is no exceptional graph.

Similarly if $n_1 = 2$, then $L(G) = L(K_{2,n_2})$ and again no exceptional graphs exist.

Further, $n_1 = 3$ implies $n_2 = 4, 5, 6$ and if $n_1 = 4$ then $n_2 = 4$. Hence, $n_1 + n_2 \leq 9$.

So, the area of exceptional graphs cospectral with regular line graphs is well determined by quite elementary means starting only from the hypothesis that they all can be produced by switching the corresponding regular line graphs.

3. Line switching

It is know that exceptional graphs are obtained by switching line graphs. However, switching line graphs can lead again to line graphs (sometimes even isomorphic to the orginal ones). Suppose $L(G_1)$ is switched with respect to a set V of its vertices and we get $L(G_2)$. The set V can be considered as a set of edges (lines) of G_1. Now, a line switching of G_1 w. r. t. V is naturally defined, which converts in to G_2. Line swiching of graph w. r. t. a set of lines V means changing positions of lines of the graph in such a way that the mutual adjacencies of lines in V and outside V remain and each line from V is adjacent to exactly those lines outside V to which it was not adjacent before switching.

Two examples of such a line switshing are given on Fig. 6. Second example shows that it is sometimes useful to consider root graphs with a number of isolated vertices.

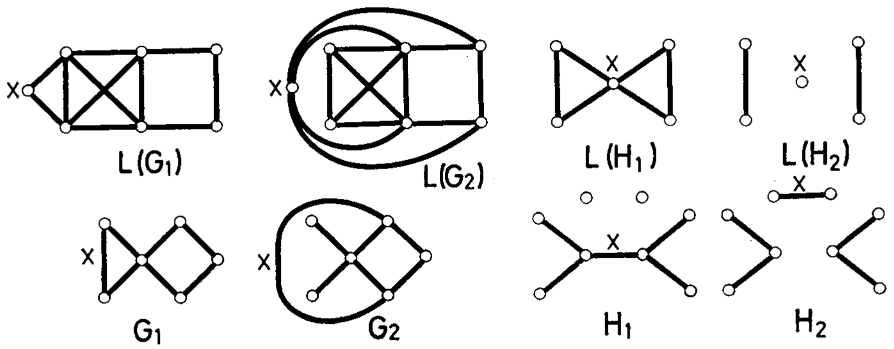

Fig. 6

An interesting effect occurs in line switching due to the fact that nonisomorphic graphs can have isomorphic line graphs. As it is known, the only such case with connected graphs occurs when one is K_3 and the other $K_{1,3}$. Such a line switching is represented in Fig. 7.

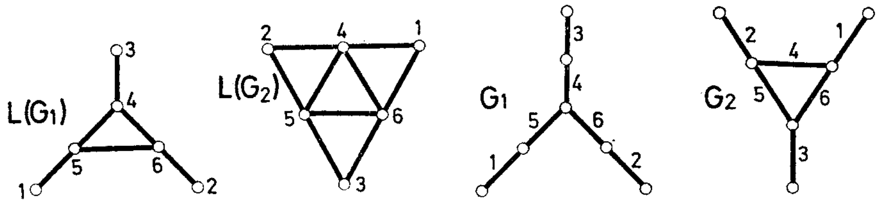

Fig. 7

Another example with two stars converted into triangles is given in Fig. 8. This example is interesting also because a regular graph is switched into a regular graphs with a different derege.

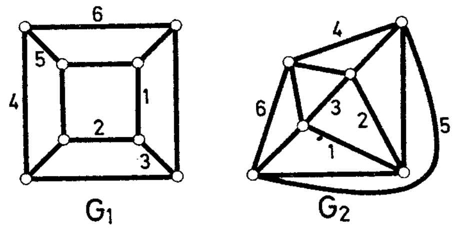

Fig. 8

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