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ON FIXED EDGES OF ANTITONE SELF-MAPPINGS OF COMPLETE LATTICES

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Abstract Studying fixed edges we start from a more general notion–p-pairs and p–points proving first that the set of all p–points of an antitone self-mapping of a complete lattice L is a sublattice of L. In this way we obtain as a direct consequence J. Klimeš's Fixed edge Theorem and provide an easy proof of his Theorem 2. Besides, this approach sheds much more light on the treated problems. In the sequel (Theorem 2) we examine under which conditions a distinguished pair (s, t) (see Notation) appearing in inconditionally complete posets is a fixed edge. In Theorem 3 the Problem in the text is solved in a special case.

In his paper [**3**] J. Klimeš introduced the concept of a *fixed* edge for the mapping of a partially ordered set (=poset) into itself and investigated conditions under which a mapping of a poset into itself has a fixed edge. For a single antitone mapping of a complete lattice into itself J. Klimeš proved the two theorems cited below.

The definition of fixed edge is as follows. Let f be a mapping of a poset P into itself and let $x \leq y$ be elements of P. An ordered pair (x, y) is a fixed edge of f if $f(x) = y$ and $f(y) = x$.

Similar pairs of points are considered in our paper [**1**] and also earlier in Kurepa's paper [**4**]. We give our definition.

Definition. Let P be a poset and $f : P \to P$. An unordered pair $\{x, y\}$ of elements of P is a $p-$*pair of* f if $f(x) = y$ and $f(y) = x$. If, in this definition, we take the ordered pair (x, y) (instead of $\{x\ y\}$) then we call it an *ordered p-pair of f*. Any member of an (ordered or unordered) p-pair of f is said to be a *p-point of f*. Fixed edges are evidently ordered p-pairs but set of all fixed edges can be a proper subset of the set of ordered p-pairs. (See the proof of Theorem 3).

THEOREM A. (A fixed edges Theorem of [**2**, Theorem 1]). *Let complete lattice and f an antitone mapping of L into itself. Then there exists a fixed edge of f. In*

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particular, (u,v) is a fixed edge of f, where

$$u = \inf\{y \in L \mid y \geq f^2(x)\}, \quad v = \sup\{x \in L \mid x \leq f^2(x)\}$$

and u is the least element in L such that (u, f(u)) is a fixed edge of f.

THEOREM B. [**2**, Theorem 2]. *Let L be a complete lattice and f an antitone mapping of L into itself. Then there exists a maximal element p in L such that* $(p, f(p))$ *is a fixed edge of f.*

Notation. We use following notation. If $f : L \to L$, then $I(f, L) = \{x \in L \mid f(x) = x\}$, and f^2 denotes $f \circ f$. Furthermore, $A = \{x \in I(f^2, L) \mid x \leq f(x)\}$, $B = \{x \in I(f^2, L) \mid x \geq f(x)\}, \sup\ A = s$, $\inf\ B = t$. If $S \subset L$, then the restriction of f to S denoted by $f \mid S$. If neither $a \leq b$, nor $b \leq a$, then we write $a \| b$.

2. THEOREM 1. *Let L be a complete lattice and* $f : L \to L$ *be an antitone mapping. Then the set of p-points of L is nonempty and is a complete lattice (a sublattice of L).*

Proof. In fact we shall prove that the set of all p-points of an antitone self-mapping f of a complete lattice L coincides with the set of all fixed points of the mapping f^2.

Since f^2 is isotone, by Tarski's theorem [**6**, Theorem 1], the set $I(f^2, L)$ is nonempty and forms a complete lattice, a sublattice of L. Let $x \in I(f^2, L)$. Then $f(x) = f^3(x) = f^2(f(x))$, and so $f(x) \in I(f^2, L)$. Moreover, $f \mid I(f^2, L)$ is permutation of $I(f^2, L)$; otherwise, for $x \neq y$ and $f(x) = f(y)$ we would have $f^2(x) = f^2(y)$, or $x = y$, which is a contradiction. For any $x \in I(f^2, L), \{x, f(x)\}$ is evidently a p-pair.

Conversely, let $\{x, y\}$ be a p-pair of f. Then $f(x) = y$ and $f(y) = x$, or $f^2(x) = x$; hence, $x \in I(f^2, L)$. Also $y \in I(f^2, L)$, which completes the proof.

Remark 1. Since evidently the ordered pair (m, M), where $m = \inf\ I(f^2, L)$, $M = \sup\ I(f^2, L)$ is a fixed edge of f, Theorem A follows.

Remark 2. Having Theorem 1 in mind the proof of Theorem B becomes very easy. Let C be a maximal chain of A, $p = \sup C$ and $x \in C$. For $y \geq x$ it follows that $f(x) \geq f(y) \geq y$; so, $f(x)$ is an upper bound for C. Hence, $p \leq f(x)$. Applying f we obtain $x \leq f(p)$, for all $x \in C$, implies $p \leq f(p)$, or $p \in C$. The maximality of p follows from the maximality of C, and Theorem B is proved.

THEOREM 2. (i) *The ordered pair (s, t) (see Notation) is an ordered p-pair, but need not be a fixed edge of f.*

(ii) *The ordered pair (s, t) is a fixed edge of f if and only if* $s \in A$.

Proof. (i) Since f is a permutation of $I(f^2, L)$ (see the proof of Theorem 1), it follows that $f(A) = B$ and $f(B) = A$. Hence $f(s)$ is a lower bound for B, and so

$$f(s) \leq t. \tag{*}$$

Also $f(t)$ is an upper bound for A, and so

$$(**) \qquad s \leq f(t).$$

From $(*)$ it follows that $s \geq f(t)$, which together with gives $f(t) = s$, proving the first part of the theorem.

To see that (s,t) need not be a fixed edge of f, let us consider the following example. Let L be the lattice in Figure 1 and let $f : L \to L$ be defined by

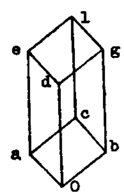
$$f = \begin{pmatrix} 0 & a & b & c & d & e & g & 1 \\ 1 & e & g & d & c & a & b & 0 \end{pmatrix}$$

Here $A = \{0, a, b\}$, $B = \{e, g, 1\}$, $\sup A = c$, $\inf B = d$, but $c \| d$, hence (c,d) is not a fixed edge of f.

(ii) If (s,t) is a fixed edge, then $s \leq t$, i.e. (by (i)) $s \leq f(s)$; so, $s \in A$.

Conversely, let $s \in A$. Then $s \leq f(s)$, or $s \leq t$ (again by (ii), proving the assertion.

Remark. The example above shows that $s \| t$ can occur. Note that it can be shown by examples that all other logical possibilities are possible, i.e.: $s < t$, $s = t$, $s > t$.



3. In this section we prove the existence of multifunctions using a more general definition of an antitone multifunction than it was done in [**3**]. (Compare [**2**] and [**5**]).

A multifunction F (or a multivalued function) from a set X to a set Y is a correspondence such that $\varnothing \neq F(x) \subset Y$ for each $x \in X$.

We say that a multifunction F on a poset X to a poset Y is antitone if, for all $x, y \in X, x \leq y$ implies:

Condition I. For each $v \in F(y)$ there exists a $u \in F(x)$ such that $v \leq u$.

Remark 1. This definition is more general than the following one, given in [**3**]:

We say that a multifunction F on a poset P into a poset Q is antitone if $x_1, x_2 \in P$, $x_1 \leq x_2$ implies $y_2 \leq y_1$ for all $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$.

The following definition is also given in [**3**].

Let $F : P \to P$ be an antitone multifunction and let x, y be elements of P. We say that the ordered pair (x, y) is a *fixed edge* of F if $x \leq y$, $x \in F(y)$ and $y \in F(x)$.

We say that a multifunction $f : P \to P$, where P is a complete lattice, is *sup-containing* if sup $F(x) \in F(x)$, for all $x \in P$.

THEOREM 3. *Let L be a complete lattice and let $F : L \to L$ be a sup-containing antitone multifunction. Then there exists a fixed edge of F.*

Proof. Let us define a single-valued mapping $f : L \to L$ by $f(x) = \sup\ F(x) \cdot f$ is well defined, since sup $F(x)$ exists for each $x \in L$. If $x \leq y$ and if we put $m = \sup\ F(y)$, then by Condition I, exists a $n \in F(x)$ such that $m \leq n$. Then $m \leq n \leq \sup\ F(x)$, or $f(y) \leq f(x)$, so that f is antitone. By Theorem 1 there exist $u, v \in L$ such that $u \leq v$, $f(u) = v$ and $f(v) = u$. It follows that $v \in F(u)$ and $u \in F(v)$ and the Theorem is proved.

Remark. Since the notion of antitone multifunction given by J. Klimeš is a special case of that given in our definition it follows that Theorem 7 in [**3**] is a corollary of Theorem 3.

4. Let L be a complete lattice and let $f : L \to L$ be an antitone self-mapping. In the set E of all fixed edges of f J. Klimeš introduced a partial order, defining $(a, b) \leq c, d)$ by ($a \geq c$ and $b \leq d$). He also showed that this set is a lattice (provided that L is a complete lattice) with a zero θ added, if necessary.

Here we shall treat the converse problem but first we shall make clear the meaning of some words.

Let P be a poset and let E be a set of ordered pairs (u, v) of elements of P ordered by: $(u, v) \leq (u', v')$ if and only if $u \geq u'$ and $v \leq v'$. Suppose that the set E (to which, if necessary, is added a zero θ) is a complete lattice, which we shall call an *edge lattice defined on P*. The set of all x's in P such that there exists an y in P such that $(x, y) \in E$, or $(y, x) \in E$, will be caled the *body of E*.

Problem. Let P be a poset and E an edge lattice on P with the body Q. Is there an antitone selfmapping of P such that $E \backslash \{\theta\}$ is the set of all fixed edges of f.

The following theorem solves this problem in a special case.

THEOREM 4. *Let P be a poset and E an edge lattice defined on P. If the body Q of E is a complete sublattice of P, Then there exists an antitone self-mapping of P such that E is the set of all fixed edges of f.*

Proof. On the body Q of E the mapping f is defined in the natural way: for all $x, y \in Q$ if $(x, y) \in E$, then $f(x) = y$ and $f(y) = x$.

Now for any $x \in P$, let $M(x) = \{z \in Q \mid z \leq z\}$. Since Q is a complete lattice sup $M(x)$ exists in Q, provided $M(x) \neq \varnothing$. Denote this supremum by s_x. For any $x \in P \backslash Q$, such that $M(x) \neq \varnothing$, we put $f(x) = f(s_x)$, where $f(s_x)$ is already defined, s_x being in Q. Let us prove that an f so defined is antitone. Take $x \leq y$. We distringuish the following four cases.

1° $x, y \in P \backslash Q$. If $z \leq x$, then $z \leq y$; hence, $\sup M(x) \leq \sup M(y)$, so that $f(x) \leq f(y)$, f being antitone on Q.

2° $x \in Q$, $y \in P \backslash Q$. Then $x \in M(y)$; hence, $x \leq \sup M(y)$ and so, $f(x) \geq f(y)$.

3° $x \in P \backslash Q$, $y \in Q$. Then $\sup\{z \in Q \mid z \leq x\} \leq y$. Also $f(x) \geq f(y)$.

4° $x, y \in P$. It is clear that $f(x) \geq f(y)$.

In this way f is defined for all $x \in P$ such that $M(x) \neq \varnothing$. If $M(x) = \varnothing$ for some $x \in P$, then we put $f(x) = \max Q$. Now f is defined on the whole set P and it is antitone, since there is no $y \in P$, $x \geq y$ with $M(x) = \varnothing$ and $M(y) \neq \varnothing$. It follows that for all $x \in P$ with $M(x) = \varnothing$ and all $y \in P$ with $M(y) = \varnothing$, $x < y$ or $x \| y$.

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