PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 34 (48), 1983, pp. 73-79

GRAPHS WITH MAXIMUM AND MINIMUM INDEPENDENCE NUMBERS

Ivan Gutman

Abstract. If $r(G,k)$ is the number of selections of k independent vertices in a graph G, and if $r(G,k) > r(H,k)$, the graph G is i-greater than the graph H. The maximal and the minimal graphs w.r.t. the above property are determined in the class of acyclic, unicyclic, connected acyclic and connected unicyclic graphs.

If G is a graph and $v_1, v_2, \dots, v_n$ are its vertices, then the vertices $v_{i_1}, \dots, v_{i_k}$ are said to be mutually independent if they pairwise non-adjancent in G. The number of ways in which k mutually independent vertices ($k \geq 2$) can be selected in G is called the *k-the independence number* of G and is denoted by $r(G,k)$. In addition, $r(G,0) = 1$ and $r(G,1) = n$ =number of vertices of G.

Let G and H be two graphs. G is *i-greater* than $H, G > H$, if $r(G,k) \geq r(H,k)$ for all values of k. If both $G > H$ and $H > G$, then G and H are said to be *i-equivalent*, $G \stackrel{i}{=} H$.

Let $\mathbf{G}$ be a set of graphs. An element G_{max} is *i-maximal* in the set $\mathbf{G}$ if $G_{max} > G$ for all $G \in \mathbf{G}$. Similarly, if $G_{min} \in \mathbf{G}$ and $G > G_{min}$ for all $G \in \mathbf{G}$, then G_{min} is called the *i-minimal* graph in the set $\mathbf{G}$.

In the present paper we determine the i-maximal and the i-minimal graphs for a number of classes of graphs.

Preliminaries

We shall use the following terminology and symobolism. $G = H$ (resp. $G \neq H$) means that the graphs G and H are (resp. are not) isomorphic. $G_1 \dot{+} G_2$ is the union of the graphs G_1 and G_2.

If the vertices v_r and v_s are adjacent in G, then the edge between them is labelled by e_{rs}. If the vertices v_r and v_s are not adjacent in G, $G + e_{rs}$ is the graph obtained from G by introducing an edge between v_r and v_s.

AMS Subject Classification (1980): Primary 05 C 35.

Let v_r be a vertex of G_1 and v_s a vertex of G_2. Then $(G_1 \dot{+} G_2) + e_{rs}$ will be denoted by $G_1(r,s)G_2$. The graph obtained by identifying the vertices v_r and v_s is $G_1[r,s]G_2$.

The complete graph, the path, the cycle and the star with n vertices are denoted by K_n, P_n, C_n and S_n, respectively. The vertices of P_n are labelled so that v_r and v_{r+1} are adjacent, $r = 1, \dots, n-1$. The vertices of C_n are labelled so that $C_n = P_n + e_{1n}$.

The graph obtained by adding k_r pendent edges to the vertex v_r of G, for all $r = 1, \dots, n$ is denoted by $G(k_1, \dots, k_n)$.

$\mathbf{F}_n$, $\mathbf{T}_n$, $\mathbf{U}_n$ and $\mathbf{U}'_n$ will denote the set of all forests, trees, unicyclic graphs and connected unicyclic graphs, respectively, with n vertices.

Some elementary properties of the independence numbers are summarized in the following lemma [**1**].

LEMMA 1. *Let G be a graph and $v_1, v_2, \dots, v_n$ be its vertices. Let A_r be the set containing v_r and all vertices adjacent to v_r. Then the following statements hold.*

(a) $r(G,k) = r(G - v_r, k) + r(G - A_r,\ k-1)$.

(b) *If the vertices v_r and v_s are not adjacent in G, then*

$$r(G,k) = r(G + e_{rs}, k) + r(G - A_r - A_s, k-2).$$

If the vertices v_r and v_s are adjacent in G, then

$$r(G,k) = r(G - e_{rs}, k) - r(G - A_r - A_s, k-2).$$

(c) $$r(G_1 \dot{+} G_2, k) = \sum_{j=0}^{k} r(G_1, j) r(G_2, k-j).$$

In order to prove the main results of this paper, namely Propositions 2–8, we need a few auxilary statements.

LEMMA 2. *If v is a vertex of G then $G > (G - v)$. If e is an edge of G, then $(G - e) > G$.*

LEMMA 3. *If $G_1 > H_1$, then for all graphs $G_2, (G_1 \dot{+} G_2) > (H_1 \dot{+} G_2)$.*

Proof. Immediate from Lemma 1 c. q.e.d.

LEMMA 4. *If G is a disconnected graph with n vertices and cyclomatic number c, then there exists a connected graph H with n vertices nod with cyclomatic number c, such that G is i-greater than H.*

Proof. If $G = G_1 \dot{+} G_2$, v_r is a vertex of G_1 and v_s is a vertex of G_2, then it is sufficient to choose $H = G_1(r,s)G_2$. q.e.d.

In [**1**] the following result has been proved.

PROPOSITION 1. *$n\ K_1$ is the i-maximal graph and P_n is the i-minimal graph in $\mathbf{F}_n$. S_n is the i-maximal graph and P_n is the i-minimal graph in $\mathbf{T}_n$.*

LEMMA 5. *Let T_n be a tree with n vertices and v_1 its vertex. If G is a graph and v_r its vertex, then $S_n(1,r)G > T_n(1,r)G > P_n(1,r)G$.*

Proof. Applying Lemma 1a to the vertex v_r of $S_n(1,r)G$, $\mathbf{T}_n(1,r)G$ and $P_n(1,r)G$, respectively, we obtain

$$\begin{aligned}
&r(S_n(1,r)G,k) = r(S_n \dot{+} (G - v_r), k) + r((n-1)K_1 \dot{+} (G - A_r), k-1),\\
&r(T_n(1,r)G,k) = r(T_n \dot{+} (G - v_r), k) + r((T - v_1) \dot{+} (G - A_r), k-1),\\
&r(P_n(1,r)G,k) = r(P_n \dot{+} (G - v_r), k) + r(P_{n-1} \dot{+} (G - A_r), k-1).
\end{aligned}$$

Lemma 5 follows now from Proposition 1 and Lemma 3. q.e.d.

LEMMA 6. *Using the same notation as in Lemma 5.*

$S_n[1,r]G > T_n[1,r]G > P_n[1,r]G$.

Proof. Let in the graph G the vertex v_r be adjacent to the vertices v_{ri}, $i = 1,\ldots,d_r$. Then by applying Lemma 1a to the vertices v_{ri}, $i = 1,\ldots,d_r$, of $S_n[1,r]G$, $T_n[1,r]G$ and $P_n[1,r]G$, we obtain

$$\begin{aligned}
&r(S_n[1,r]G,k) = r(S_n \dot{+} (G - A_r), k) + \sum_{i=1}^{d_r} r((n-1)K_1 \dot{+} (G - A_{ri}), k-1),\\
&r(T_n[1,r]G,k) = r(T_n \dot{+} (G - A_r), k) + \sum_{i=1}^{d_r} r((T_n - v_1) \dot{+} (G - A_{ri}), k-1),\\
&r(P_n[1,r]G,k) = r(P_n \dot{+} (G - A_r), k) + \sum_{i=1}^{d_r} r(P_{n-1} + (G - A_{ri}), k-1).
\end{aligned}$$

Lemma 6 follows again from Proposition 1 and Lemma 3. q.e.d.

LEMMA 7. *For $j = 1,2,\ldots,n-1$, $(P_1 \dot{+} P_{n-1}) > (P_j \dot{+} P_{n-j}) > (P_2 \dot{+} P_{n-2})$.*

Proof. We use induction on the number n. The validity of Lemma 7 is easily checked for $n \le 8$.

We suppose now that Lemma 7 holds for $n = h-2$ and $n = h-1$, and deduce its validity for $n = h$. It is legitimate to assume that $h > 8$.

By Lemma 1a, $r(P_j \dot{+} P_{h-j}, k) = r(P_j \dot{+} P_{h-1-j}, k) + r(P_j \dot{+} P_{h-2-j}, k-1)$, which combined with the hypothesis

$$r(P_1 \dot{+} P_{h-3}, k-1) \ge r(P_j \dot{+} P_{h-2-j}, k-1) \ge r(P_2 \dot{+} P_{h-4}, k-1)$$

and $r(P_1 \dot{+} P_{h-2}, k) \ge r(P_j \dot{+} P_{h-1-j}, k) \ge r(P_2 \dot{+} P_{h-3}, k)$ yields the required statement for $n = h$. q.e.d.

LEMMA 8. *For $1 < j < n-1$, $P_{n-1}(2,1)P_1 > P_{n-1}(j,1)P_1 > P_{n-1}(3,1)P_1$.*

Proof. By Lemma 1a, $r(P_{n-1}(j,1)P_1,k) = r(P_{n-1},k) + r(P_{j-1} + P_{n-1-j}, k-1)$. Since the first term on the r.h. s. is independent of j, we obtain Lemma 7. q.e.d.

Lemma 9. *For* $1 < j < n-2$, $P_{n-2}(2,1)P_2 > P_{n-2}(j,1)P_2 > P_{n-2}(3,1)P_2$.

Proof. Analogous, and based on the relation

$$r(P_{n-2}(j,1)P_2,k) = r(P_{n-2}(j,1)P_1,k) + r(P_{n-2},k-1)$$

and on Lemma 8. q.e.d.

Lemma 10. $P_{n-1}(3,1)P_1 > P_{n-2}(3,1)P_2$.

Proof. Lemma 10 can be proved by induction on n, using the fact that

$$r(P_a(i,j)P_b,k) = r(P_{a-1}(i,j)P_b,k) + r(P_{a-2}(i,j)P_b,k-1).$$ q.e.d.

Let v_r and v_s the two adjacent vertices of a graph G. The substitution of the edge e_{rs} by a path with a vertices yields the graph $G(e_{rs} \mid a)$; see Fig. 1.

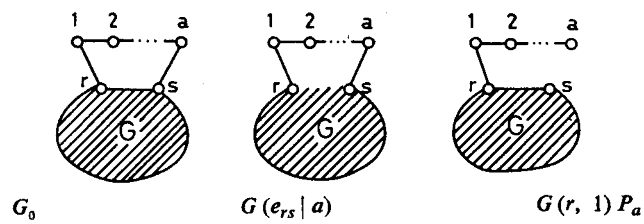

Fig. 1

Lemma 11. $G(r,1)P_a > G(e_{rs} \mid a)$.

Proof. Let the vertices of the graphs $G_0, G(e_{rs} \mid a)$ and $G(r,1)P_a$ be labelled as indicated in Fig. 1. Then $G(r,1)P_a + e_{as} = G(r_{rs} = G(e_{rs} \mid a) + re_{rs} = G_0$. According to Lemma 1b,

$$r(G(r,1)P_a,k) = r(G_0,k) + r((G - A_s) \dotplus P_{a-2}, k-2),$$
$$r(G(e_{rs} \mid a),k) = r(G_0,k) + r((G - A_r - A_s) \dotplus P_{a-2}, k-2).$$

$G - A_r - A_s$ is an induced subgraph of $G - A_s$. Therefore by Lemma 2, $r(G - A_s,k) \geq r(G - A_r - A_s, k)$, where as by Lemma 3, $r((G - A_s) \dotplus P_{a-2},k) \geq r((G - A_r - A_s) \dotplus P_{a-2},k)$ for all values of k. Lemma 11 now follows from the two equalities above q.e.d.

Lemma 12. (a) *For* $3 \leq j \leq n$, $C_j(1,1)P_{n-1} \stackrel{i}{=} C_{n-j+3}(1,1)P_{j-3}$.

(b) *For* $3 < j < n$, .

$$C_4(1,1)P_{n-4} \stackrel{i}{=} C_{n-1}(1,1)P_1 > C_j(1,1)P_{n-j} > C_5(1,1)P_{n-5} \stackrel{i}{=} C_{n-2}(1,1)P_2.$$

Proof. By Lemma 1a,

$$r(C_j(1,1)P_{n-j},k) = r(P_{n-1},k) + r(P_{j-3} \dot{+} P_{n-j}, k-1),$$

from which it follows that $C_j(1,1)P_{n-j}$ and $C_{n-j+3}(1,1)P_{j-3}$ are i-equivalent.

In addition, because of Lemma 7, the r.h.s. of the above equality will be maximal if $j-3=1$ or $n-j=1$; the same expression will be minimal for $j-3=2$ or $n-j=2$. q.e.d.

The main results

PROPOSITION 2. *$(n-2)K \dot{+} K_2$ is the i-maximal graph and $P_{n-2}(3,1)P_2$ is the i-minimal graph in the set $\mathbf{F}_n \backslash \{nK_1, P_n\}$.*

Proof. The first part of Proposition 2 is evident.

Let F be the i-minimal graph in $\mathbf{F}_n \backslash \{nK_1, P_n\}$. Then F must be connected (because of Lemma 4), it must have exactly one vertex of degree greater than two (because of Lemma 5) and this vertex must be of degree three (because of Lemma 6). Let v be a vertex of F having degree one and let v be adjacent to w. Then $F-v \in \mathbf{T}_{n-1}$, $F-v-w \in \mathbf{F}_{n-2}$ and

$$r(F,k) = r(F-v,k) + r(F-v-w,\, k-1).$$

Now because of Proposition 1, F can be i-minimal only if (a) $F-v = P_{n-1}$ and /or (b) $F-v-w = P_{n-2}$.

In case (a) we have $F = P_{n-1}(j,1)P_1$. If $j=1$ or $j=n$, then $F=P_n$, which is impossible. If $1<j<n$, then F is not i-minimal because of Lemmas 8 and 10. Hence case (a) leads to contradictions.

In case (b), $F = P_{n-2}(j,1)P_2$. It must be that $1<j<n$; otherwise $F=P_n$. But then, because of Lemma 9, $F = P_{n-2}(3,1)P_2$. q.e.d.

PROPOSITION 3. *$P_2(n-3,1)$ is the i-maximal graph and $P_{n-2}(3,1)P_2$ is the i-minimal graph in the set $\mathbf{T}_n \backslash \{S_n, P_n\}$.*

Proof. Having in mind Proposition 2, one has to prove only that $P_2(n-3,1)$ is the i-maximal graph.

Let T be any element of $\mathbf{T}_n \backslash \{S_n, P_n\}$, and let v_r be its vertex of degree one. Then $r(T,k) = r(T-v_r,k) + r(T-A_r,k-1)$

with $T-v_r \in \mathbf{T}_{n-1}$ and $T-A_r \in \mathbf{F}_{n-2}$. In order to have a maximal value for $r(T,k)$ we have to choose $T-v_r = S_{n-1}$ (because of Proposition 1) and $T-A_r = (n-4)K_1 \dot{+} K_2$ (because of Proposition 2). This, on the other hand, implies $T = P_2(n-3,1)$ q.e.d.

Using similar considerations one proves

PROPOSITION 4. *$(n-3)K_1 \dot{+} S_3$ is the i-maximal graph in the set $\mathbf{F}_n \backslash \{nK_1, (n-2)K_1 \dot{+} K_2\}$. $P_2(n-4,2)$ is the i-maximal graph in the set $\mathbf{T}_n \backslash \{S_n, P_2(n-3,1)\}$.*

Proposition 5. *$(n-3)K_1 \dot{+} C_3$ is the i-maximal graph in the set $\mathbf{U}_n$. C_n and $C_3(1,1)P_{n-3}$ are two (mutually i-equivalent) i-minimal graphs in the set $\mathbf{U}_n$.*

Proof. Every graph $U \in \mathbf{U}_n$ contains a vertex v_r whose degree is greater than one, such that $U - v_r$ is a forest with $n-1$ vertices and with at least one edge, where as $U - A_r$ is a forest with at most $n-3$ vertices. Now, by Proposition 2, $((n-3)K_1 \dot{+} K_2) > (U - v_r)$ and by Proposition 1 and Lemma 2, $(n-3)K_1 > (U - A_r)$. It is easy to see that $U = (n-3)K_1 \dot{+} C_3$ is the unique graph with the properties $U - v_r = (n-3)K_1 \dot{+} K_2$ and $U - A_r = (n-3)K_1$. This proves the first part of Proposition 5.

The fact that C_n is i-minimal in $\mathbf{U}_n$ is an immediate consequence of Lemma 11. Then the second part of Proposition 5 follows from Lemma 12. q.e.d.

A similar reasoning leads also to

Proposition 6a. *$(n-4)K_1 \dot{+} C_4$ and $(n-4)K_1 \dot{+} C_3(1,0,0)$ are the two (mutually i-equivalent) i-maximal graphs in the set $\mathbf{U}_n \backslash \{(n-3)K_1 \dot{+} C_3\}$.*

Proposition 7. *$C_3(n-3,0,0)$ is the i-maximal graph, whereas C_n and $C_3(1,1)P_{n-3}$ are the two (mutually i-equivalent) i-minimal graphs in the set $\mathbf{U}'_n$.*

Proof. Having in mind Proposition 5, only the first part of Proposition 7 remains to be proved. Let U be the i-maximal element of $\mathbf{U}'_n$ and let q be the size of its cycle. By Lemma 6, U must be of the form $C_q(k_1, k_2, \dots, k_q)$, where $k_i \geq 0$.

Let v_r be a vertex of degree one of the graph U. Then $U - v_r \in \mathbf{U}'_{n-1}$ and $U - A_r \in \mathbf{F}_{n-2} \backslash \{(n-2)K_1\}$.

We complete the proof by induction on the number of vertices of U. For $n = 4, 5$ and 6 it can be established easily that $U = C_3(n-3,0,0)$. Suppose now that $C_3(h-4,0,0)$ is i-maximal in $\mathbf{U}'_{h-1}$. Then

$$r(C_3(h-3,0,0),k) = r(C_3(h-4,0,0),k) + r((h-4)K_1 \dot{+} K_2, k-1).$$

Since by Proposition 2, $(h-4)K_1 \dot{+} K_2$ is i-maximal in $\mathbf{F}_{h-2} \backslash \{(h-2)K_1\}$, we conclude that $C_3(h-3,0,0)$ is i-maximal in $\mathbf{U}'_h$. q.e. d.

Analogous considerations also lead to

Proposition 8a. *$C_3(n-4,1,0)$ is the i-maximal graph in the set $\mathbf{U}'_n \backslash \{C_3$ $(n-3,0,0)\}$.*

In order to complement the results exposed in Proposition 6a and 8a, we determine also the second i-minimal unicyclic graphs.

Propositions 6b and 8b. *The two i-equivalent graphs $C_5(1,1)P_{n-5}$ and $C_{n-2}(1,1)P_2$ are i-minimal in the set $\mathbf{U}_n \backslash \{C_n, C_3(1,1)P_{n-3}\}$ (and therefore also in the set $\mathbf{U}'_n \backslash \{C_n, C_3(1,1)P_{n-3}\}$).*

Proof. The i-minimal graph in $\mathbf{U}_n \backslash \{C_n, C_3(1,1)P_{n-3}\}$ must be connected (because of Lemma 4) and must possess exactly one vertex of degree one (because of Lemmas 5 and 11). Hence this graph must be of the form $C_j(1,1)P_{n-j}$. The rest of the proof follows from Lemma 12. q.e. d.

Acknowledgment. The author thanks Professor Horst Sachs (Ilmenau, GDR) for useful discussions.

REFERENCES

[1] I. Gutman, F. Harary, *Generalizations of the matching polynomial*, Utilitas Math., **24** (1983), 97–106.

Prirodno-matematički fakultet
p.p. 60
34001 Kragujevac
Jugoslavija

(Received 12 11 1982)