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FORMULAE FOR GENERAL REPRODUCTIVE SOLUTIONS OF CERTAIN MATRIX EQUATIONS

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Abstract. The formulae for general reproductive solutions of certain matrix equations are given. They contain generalized inverse of the given square (respectively rectangular) matrix and one arbitrary matrix of the corresponding form connection with it four theorems and a corollary are given.

1. In this paper we give some formulae for general reproductive solutions of certain matrix equations. We consider as especially interesting the equation (1) in the following definition.

*Definition*1. [**1**] Let A be a $m \times n$ matrix over a certain field. Then a $n \times m$ matrix X that satisfies any or all of Penrose's equations:

$$AXA = A, \tag{1}$$

$$XAX = X, \tag{2}$$

$$(AX)^* = AX, \tag{3}$$

$$(XA)^* = XA, \tag{4}$$

is called a generalized inverse of a given matrix A.

We say that X is a $\{1\}$-inverse of a matrix A if it satisfies equation (1), a $\{1,2\}$-inverse of matrix A if it satisfies both equations (1) and (2) etc. Accordingly, the set of all $\{1\}$-inverses of a matrix A will be denoted by $A\{1\}$, the set of all $\{1,2\}$-inverses by $A\{1,2\}$ etc.

A matrix X which satisfies all four equations is called the Moore-Penrose inverse of A and is usually denoted by A^+.

We have the following definition ([**6**] and [**7**]).

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Definition 2. Let $J(x)$ be an equation possible in $x(x$ is a member of a given set $S)$. A formula of the form

$$x = f(\lambda), \tag{5}$$

where $f : S \to S$, is called a general reproductive solution of $J(x)$ if the following conditions are satisfied:

(i) Formula (5) is the formula of general solution of the equation $J(x)$, i.e., the following implications hold

$$(\forall \lambda,\ x \in S)\ \ (x = f(\lambda) \Rightarrow J(x))$$
$$(\forall x \in S)(J(x) \Rightarrow (\exists \lambda \in S)\ x = f(\lambda)).$$

(ii) Besides the last implication the following implication (condition of reproduction) holds

$$J(x) \Rightarrow x = f(x).$$

It is interesting that the two implications (i) are equivalent to one equivalence

$$(\forall x \in S)(J(x) \Leftrightarrow (\exists \lambda \in S)\, x = f(\lambda)). \tag{6}$$

In [**6**] it is proved that for each equation $J(x)$, possible in x, a formula of general reproductive solution exists. Indeed, assume that (5) is a formula of general solution of the equation $J(x)$, i. e., that the implications (i) hold. Further, let the second implication in (i) hold if $\lambda = g(x)$, where $g : S \to S$, i. e., let the implication

$$(\forall x \in S) J(x) \Rightarrow x = f(g(x))$$

holds. It is easy to see that in this case the formula

$$x = f(g(\lambda)) \tag{7}$$

is a formula of general reproductive solution of the equation $J(x)$. In this paper we shall use this idea of "making reproductive solutions" in order to find the formulae of general reproductive solution for certain important matrix equations.

2.1. In [**5**] S. B. Prešić has constructed a matrix B, for any square matrix A, for which $ABA = A$, ensuring thus the existence of a $\{1\}$-inverse for each square matrix. In the same paper he proved the following theorem.

THEOREM A.[**5**] *Let A be a given square matrix of order n and of rank $r \le n$ with elements from any field K, and let $B \in A\{1\}$. Then*

1° $AX = 0 \Leftrightarrow (\exists Q \in K^{n\times m})X = (I - BA)Q,$

2° $XA = 0 \Leftrightarrow (\exists Q \in K^{m\times n})X = Q(I - AB),$

3° $AXA = A \Leftrightarrow (\exists Q \in K^{n\times n})X = B + Q - BAQAB,$

4° $AX = A \Leftrightarrow (\exists Q \in K^{n\times n})X = I + (I - BA)Q,$

5° $XA = A \Leftrightarrow (\exists Q \in K^{n\times n})X = I + Q(I - AB),$

where $K^{m\times n}$ denotes the class of all $m \times n$ matrices over a field K.

Remark 1. The ($\Leftarrow$)-parts of this theorem are proved by inspection. In the ($\Rightarrow$)-parts for Q, as a function of X, we take respectively $X, X, X-B, X+BA-I$, $X+AB-I$.

2.2. The right-hand side of each of the equivalences in Theorem A is the formula of general solution of the corresponding equation on the left-hand side. Mean-while, it is easy to prove the following modified theorem in which the right-hand side of each of the stated equivalences is the formula of general reproductive solution of the corresponding equation from the left-hand side.

THEOREM 1. *Let A be any given square matrix of order n and rank $r \le n$ with elements from an arbitrary field K, and let $B \in A\{1\}$. Then*

$(1^\circ)'$ $\quad AX = 0 \Leftrightarrow (\exists Q \in K^{n\times m}) X = (I - BA)Q$,

$(2^\circ)'$ $\quad XA = 0 \Leftrightarrow (\exists Q \in K^{m\times n}) X = Q(I - AB)$,

$(3^\circ)'$ $\quad AXA = A \Leftrightarrow (\exists Q \in K^{n\times n}) X = Q + BAB - BAQAB$,

$(4^\circ)'$ $\quad AX = A \Leftrightarrow (\exists Q \in K^{n\times n}) X = Q + BA - BAQ$,

$(5^\circ)'$ $\quad XA = A \Leftrightarrow (\exists Q \in K^{n\times n}) X = Q + AB - QAB$.

Formulae of general reproductive solution in Theorem 1 can be obtained using (7) and Remark 1: in $3^\circ, 4^\circ$ and 5° in Theorem A we replace Q by $(Q-B)$, $(Q+BA-I)$ and $(Q+AB-I)$ respectively.

It is especially interesting that in both of the stated theorems the right-hand side of equivalence 3° (respectively $(3^\circ)'$) gives the characterization of the set $A\{1\}$ of a given square matrix. This is stated in the following corollary.

COROLLARY 1. *Let $A \in K^{n\times n}$ be of rank $r \le n$ and $B \in A\{1\}$. Then*

$$A\{1\} = \{B + Q - BAQAB : Q \in K^{n\times n}\}, \tag{8}$$

respectively

$$A\{1\} = \{Q + BAB - BAQAB : Q \in K^{n\times n}\}. \tag{8'}$$

2.3. We can note in particular that the formula of general solution of the equation $AXA = A$ in Theorem A can at the same be reproductive, if for B we do not take any $\{1\}$-inverse, but some of the $\{1,2\}$-inverses of a given matrix A. Such a $\{1,2\}$-inverse exists. Namely, it is known [**1**] that each $\{1\}$-inverse $A^{(1)}$ for which rank $A^{(1)}$ =rank A is a $\{1,2\}$-inverse of the matrix A.

THEOREM 2. *For a given matrix $A \in K^{n\times n}$ and an arbitrary matrix $B \in A\{1,2\}$ the formula of general reproductive solution of the equation* (1) *is*

$$X = B + Q - BAQAB; \quad Q \in K^{n\times n}. \tag{9}$$

Indeed if $B \in A\{1,2\}$ then for $Q = X_0$, where $AX_0A = A$, we get $B + Q - BAQAB = B + X_0 - BAX_0AB = B + X_0 - BAB = B + X_0 - B = X_0$ by which reproductivity is proved.

3. Similarly, as in the case of a square matrix A we can extend the above results to the rectangular matrices.

The existence of the $\{1\}$-inverse of a rectangular $m \times n$ matrix has been shown by A.G. Fischer in [**3**]. He constructed such an inverse by transforming the given matrix into a special Hermite normal form, although it is possible to find this inverse as in [**5**].

So we have the following two theorems.

THEOREM 3. *Let $A \in K^{m\times n}$ and $A^{(1)} \in A\{1\}$. Then the formula of general reproductive solution of the equation $AXA = A$ is*

$$X = Y + A^{(1)}AA^{(1)} - A^{(1)}AYAA^{(1)}, \tag{10}$$

where Y is an arbitrary $n \times m$ matrix with elements from the field K.

THEOREM 4. *Let $A \in K^{m\times n}$, and $A^{(1,2)} \in A\{1,2\}$. Then the formula of general reproductive solution of the equation $AXA = A$ is*

$$X = A^{(1,2)} + Y - A^{(1,2)}AYAA^{(1,2)}, \tag{11}$$

where Y is an arbitrary $n \times m$ matrix with elements from the field K.

Proofs are omitted, since they are similar to those already given.

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