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LINEARIZATION OF NONLINEAR DIFFERENTIAL EQUATIONS, IV: NONLINEAR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS EQUIVALENT TO LINEAR BASE EQUATION

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1. Introduction. The linearization as a method for solving some nonlinear differential equations (partial and ordinary) is well known in the theory of differential equations. Already Painlevé, in his classical papers [**1, 2, 3**] quoted several examples of linearization of nonlinear differential equations; Kamke [**4**], lists over twenty nonlinear ordinary differential equations whose solutions are expressed as function of solutions of the corresponding linear equations; by classical transformations such as Kirchoff's, Bäcklund's, some nonlinear partial equations are reduced to linear ones (see, for example [**6**]). However, only upon the appearance of Pinney's note [**7**] in 1950, this topic developed suddenly. This note was the starting point for numerous investigations these last years [**8-24**].

In several papers [**15-24**] the authors investigated the problems of the following type:

Construct a nonlinear differential equation of the form

$$\Phi(z_{x_1x_1},\dots,z_{x_nx_n},z_{x_1},\dots,z_{x_n},z,h(u,v)W^2,x_1,\dots,x_n)=0, \tag{1.1}$$

where u,v are particular solutions of the linear equation with functional coefficients:

$$Lu=\sum_{\substack{i,j=1\\ i\le j}}^{n}A_{ij}u_{x_ix_j}+\sum_{i=1}^{n}B_iux_i+Cu=0, \tag{1.2}$$

W is a generalization of the Wronskian (see, for example, [**17**]) defined by

$$W^2=\sum_{\substack{i,j=1\\ i\le j}}^{n}A_{ij}(v_{x_i}u-u_{x_i}v)(v_{x_j}u-u_{x_j}y), \tag{1.3}$$

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h is given homogenous function in two variables, having the solution in the form

$$z = F(u,v) \quad (F \text{ is given function}). \tag{1.4}$$

The linear eqaution (1.2) is called the "base equation".

For different functions F, which for all investigated cases were homogeneous, the equations of the type (1.1) were constructed. Particularly, the case of ordinary differential equations was investigated [**18-21, 24**], as well as the case of partial [**15–17, 22, 23**]. The results were also applied for solving some problems of quantum mechanics, thermodynamics, etc.

Since in all above mentioned papers only the particular cases were investigated, naturally the following general problem arises: Determine the necessary and sufficient conditions so that the equation.

$$z_{x_1x_1} = f(z_{x_1x_2},\dots,z_{x_nx_n},z_{x_1},\dots,z_{x_n},z,h(u,v)w_1,\dots,h(u,v)w_n,x_1,\dots,x_n) \tag{1.5}$$

has the solution (1.4), where u, v are linearly independent particular solutions of linear equation (1.2),homogeneous function of two variables of order $k, w_i = v_{x_i}u - u_{x_i}v (i = 1,\dots,n)$.

Remark 1. Equation of the form (1.5) is more general than (1.1). The assumption that $z_{x_1x_1}$ appears of the left hand side is taken due only to technical reasons and the generality is not decreased. In connection with that we assume $A_{11} = -1$. Also, we suppose that functions F and f have the continuous third order derivatives, which, having in view the nature of the problem, is justified.

In further text we shall use the following notations:

$$\sum_{i,j}{}' a_{ij} = \sum_{\substack{i,j=1\\ i\le j}}^{n} a_{ij},\ \sum_{i.j} a_{ij} = \sum_{i,j=1}^{n} a_{ij}, \sum_i a_i = \sum_{i=1}^{n} a_i.$$

In the present paper we shall give the complete solution of the posed problem. Also, we give a number of applications of the main theorem. Our result is compared to many known results.

2. The main theorem. THEOREM 1. *Equation* (1.5) *has the solution* $z = F(u,v)$, *where* u, v *are arbitrary linearly independent particular solutions of* (1.2), *if and only if it has the form:*

$$\sum_{i,j}{}' A_{ij}(z_{x_ix_j} + P(z)z_{x_i}z_{x_j}) + \sum_i B_i z_{x_i} + CQ(z) + DR(z) = 0, \tag{2.1}$$

where P, Q, R *satisfy the conditions*

$$1 - PQ = Q',\ R'Q + (2k + 3 + PQ)R = 0, \tag{2.2}$$

A_{ij}, B_i, C, D *are functions of* $x_1,\dots,x_n$ *and*

$$D = KW^2h(u,v)^2, \quad (K = \text{const.}). \tag{2.3}$$

In this case the function F is determined by

$$Q(F)/R(F) = (uG(v/u))^{2k+4}, \tag{2.4}$$

where G satisfies the ordinary differential equation

$$G''(t) + KH(t)^2G(t)^{-2k-3} = 0 \tag{2.5}$$

and H is given by

$$h(u,v) = u^k H(v/u). \tag{2.6}$$

Proof. We shall give only an outline of the proof. First, we assume that $z = F(u,v)$ satisfies (1.5), where u, v are arbitrary linearly independent particular solutions of (1.2). Than the following condition must be satisfied:

(2.7)

$$\begin{aligned}
&\sum_{\substack{i,j\\ i\neq 1}}{}' A_{ij}(F_u u_{x_ix_j}) + F_v v_{x_ix_j} + \sum_i B_i(F_u u_{x_i} + F_v v_{x_i})\\
&+ C(F_u u + F_v v) + F_{uu}u_{x_1}^2 + 2F_{uv}u_{x_1}vx_1 + F_{vv}vx_1^2\\
&= f((F_u u_{x_1x_2} + F_v v_{x_1x_2} + F_{uu}u_{x_1}u_{x_2}) + F_{uv}(u_{x_1}v_{x_2} + u_{x_2}v_{x_1} + F_{vv}v_{x_1}v_{x_2}),\\
&\ldots, (F_u u_{x_nx_n} + F_v v_{x_nx_n} + F_{uu}u_{x_n}^2 + 2F_{uv}u_{x_n}v_{x_n} + F_{vv}v_{xn}^2),\\
&(F_u u_{x_1} + F_v v_{x_1}), \ldots, (F_u u_{x_n} + F_v v_{x_n}),\ F, h(u,v)w_1, \ldots, h(u,v)w_n, x_1, \ldots, x_n).
\end{aligned}$$

Differentiating (2.7) twice with respect to variables $u_{x_ix_j}$ and $v_{x_ix_j}(i \leq j; i,j = 1, \ldots, n)$ we find that f has the form

$$\begin{aligned}
&f(s_{1,2}, \ldots, s_{nn}, s_1, \ldots, s_n, t_1, \ldots, t_n, s, x_1, \ldots, x_n)\\
&= \sum_{\substack{i,j\\ i\neq j}}{}' A_{ij}s_{ij} + p(s_1, \ldots, s_n, t_1, \ldots, t_n, s, x_1, \ldots, x_n).
\end{aligned} \tag{2.8}$$

Furthermore, substituting f, given by (2.8), into (2.7) we obtain the condition which must be satisfied by p. From the second derivative with respect to variables $u_{x_i}, v_{x_j}(i,j = 1, \ldots, n)$, from this relation we find that p has the form:

$$\begin{aligned}
p(s_1, \ldots, s_n, t_1, \ldots, t_n, s, x_1, \ldots, x_n) = \sum_{i,j}{}'(Q_{ij}^1 s_i s_j + Q_{ij}^3 t_i t_j) +\\
+ \sum_{i,j} Q_{ij}^2 s_i t_j + \sum_i (R_i^1 s_i + (R_i^2 t_i) + T,
\end{aligned}$$

where the functions $Q_{ij}^1,\ Q_{ij}^2,\ Q_{ij}^3,\ R_i^1,\ R_i^2,\ T$ depend on $s, x_1, \ldots, x_n$.

Substituting so obtained p into (2.7) and (2.8) we find the following:

$$Q_{ij}^1 = A_{ij}P(s), Q_{ij}^2 = R_i^2 = 0,\ Q_{ij}^3 = A_{ij}KR(s),\ R_i^1 = B_i,\ T = CQ(s),$$

$(i, j = 1, \ldots, n)$, where P, Q, R satisfy (2.2). Also, we have that F is determined by (2.4), where (2.5) and (2.6) hold. This proves that the mentioned conditions are necessary.

By a direct varification we can show that the same conditions are sufficient which completes the proof of the theorem.

3. Corollaries. Now, we shall give certain corollaries and examples which illustrate the theorem.

(i) Transformation $Z = (Q(z)/R(z))^{1/(2k+4)}$ reduces (2.1) to the equation

$$LZ + DZ^{-2k-3} = 0, \tag{3.1}$$

which has the solution $Z = uG(v/u)$, G satisfies (2.5), (2.6).

Equation (3.1) represents a "canonical equation" for this class of problems. Namely, the main result can be formulated in the following way: the only equations of the form (1.5) which have the solution $z = F(u, v)$, are those obtained from (3.1) by a transformation of unknown function $Z = \varphi(z)$.

(ii) Equation

$$\sum_{i,j}{}' A_{ij}(z_{x_i x_j} + (a-1) z_{x_i} z_{x_j}/z) + \sum_i B_i z_{x_i} + Cz/a + Dz^q/a = 0, \tag{3.2}$$

$(q = -2(k+2)a + 1,\ a = \text{const.})$, is a special case of (2.1) and has the solution $z = (uG(v/u))^{1/a}$, G satisfies (2.5), (2.6).

Some particular cases of the above equation are studied in [**15-17, 22, 23**].

(iii) Specially, if

$$h(u,v)^2 = u^{2k}\Big(\sum_{i=1}^m a_i (v/u)^{b_i}\Big)^r \Big(\sum_{i,j=1}^m c_{ij}(v/u)^{d_{ij}}\Big), \tag{3.3}$$

$$r = 2p(k+2) - 2,\ c_{ij} = p a_i a_j b_i (1 - b_i + (1-p) b_j),\ d_{ij} = b_i + b_j - 2,\ p, a_i, b_i$$

are constants, then equation (3.2) has the solution

$$z = u^{1/a}\Big(\sum_{i=1}^m a_i (v/u)^{b_i}\Big)^{p/a}.$$

The special cases $m = 1, 2, 3$ are treated in [**15-17, 22, 23**].

(iv) Equation (2.1), where D is given by

1° $D = kW^2(C_1u^2 + C_2uv + C_3v^2)^k$, $C_1, C_2, C_3 = \text{const}$;

2° $D = aW^2u^{2k-b}v^b$, $a, b = \text{const}$;

3° $D = aW^2u^{2k}$, $a = \text{const}$;

4° $D = aW^2u^kv^k$, $a = \text{const}$,

has the following solutions:

1° $Q(z)/R(z) = t(v/u)^{k+2}\Big(uA\Big(\int(t(v/u)^{-1}d(v/u)\Big)\Big)^{2k+4}$

where $t(v/u) = C_1 + C_2v/u + C_3(v/u)^2$, and A is determined by

$$\tag{3.4} \begin{cases} \displaystyle\int (c + C_4A^2 + C_5A^{-2k-2})^{-1/2}dA = s + d, \ k \neq -1, \\ \displaystyle\int (c + C_4A^2 - 2K \log\, A)^{-1/2}dA = s + d, \ k = -1, \end{cases}$$

$C_4 = C_2^2/4 - C_1C_3,\ C_5 = K/(k+1),\ c, d$ are arbitrary constants.

In this case F containes two arbirtary constants. Specially, for $c = 0$, the solution can be expressed in closed form

2° $Q(z)/R(z) = (4a(k+2)^2/(b+2)(2k+2-b))u^{2k+2-b}v^{b+2}$.

3° $Q(z)/R(z) = u^{2k+4}A(v/u)^{2k+4}$, where $A(s)$ is determined by (3.4), with $C_1 = 1, C_2 = C_3 = C_4 = 0$.

4° $Q(z)/R(z) = u^{k+2}v^{k+2}A(\log(v/u))^{2k+4}$, $A(s)$ is given by (3.4), with $C_1 = C_3 = 0,\ C_2 = 1,\ C_4 = 1/4$.

(v) Equations

1° ${\sum'}_{i,j} A_{ij}(z_{x_ix_j} + az_{x_i}z_{x_j}) + \sum\limits_i B_iz_{x_i} + C/a + De^{-(2k+4)az}/a = 0$

2° ${\sum'}_{i,j} A_{ij}(z_{x_ix_j} - z_{x_i}z_{x_j}/z) + \sum\limits_i B_iz_{x_i} + Cz\log\, z + Dz(\log\, z)^{-2k-3} = 0,$

3° $Lz + KW^2z^{-3} = 0,$

4° $Lz - W^2(C_1u^2 + C_2uv + C_3v^2)^{-3}z^3 = 0,$

where a, C_1, C_2, C_3 are constants and D is given by (2.4), have the solutions:

1° $z = \log(uG(v/u))/a,\ G$ satisfies (2.5), (2.6),

2° $z = \exp(uG(v/u)),\ G$ satisfies (2.5), (2.6),

3° $z = (cu^2 + duv + (d^2 - K)v^2/4c)^{1/2},\ c,\ d$ arbitrary constants,

4° $z = u(C_1 + C_2v/u + C_3v^2/u^2)^{1/2}A\Big(\int(C_1 + C_2v/u + C_3v^2/u^2)^{-1}d(v/u)\Big),$

where A is determined by $\int(c + C_4A^2 + A^4/2)^{1/2}dA = s + d(c,\ d$ are arbitrary constants). In this case A is expressed by elliptic functions.

4. The case when u and v are functionally dependent. We shall consider the case when u and v are connected by the relation:

$$\tag{4.1} v = g(u)u,$$

where g is a twice differentiable function.

Since v satisfies (1.2), we find that u, besides (1.2) must satisfy the first order partial differential equation

$$\tag{4.2} \sum_{i,j}{}' A_{ij}u_{x_i}u_{x_j} = Cu^2g'(u)\Big(ug''(u) + 2g'(u)\Big)^{-1} = CT(u).$$

Also, function D has the form

$$D = KCu^{2k+6}H\Big(g(u)^2g'(u)^3(ug''(u)+2g'(u)\Big)^{-1}. \tag{4.3}$$

Furthermore, let us suppose that C and D satisfy the following

$$D = LC\,(L = \text{const.}). \tag{4.4}$$

Then, equation (2.1), where D is given by (4.3), has the solution

$$Q(z)/R(z) = \Big(uG(g(u))\Big)^{2k+4}, \tag{4.5}$$

where G satisfies the equation

$$G''(t) + B(t)G(t)^{-2k-3} = 0, \tag{4.6}$$

and B is determined by

$$B\Big(g(u)\Big) = L\Big(ug''(u)+2g'(u)\Big)g'(u)^{-3}u^{-2k-6}, \tag{4.7}$$

u is a common particular solution of (1.2) and (4.2).

Equations of the form (2.1) with (4.4) are very interesting due to numerous applications, particulary in physics.

In papers [**15-17, 22**] only the case $v = 1/u$ was treated.

5. Equations with constant coefficients. Let us study the equation (2.1) where A_{ij}, B_i, C, D are constants $(i,j = 1,\dots,n;\ i \le j)$. Since $D = LC(L = \text{const.})$ we can apply the result from the previous chapter. In connection with that the following question rises: When, for given g, there exists u which satisfies linear equation (1.2) and first order equation (4.2)?

In some special cases the answer to this question is affirmative. For example, let us consider the case $g(u) = u^a$, $a = \text{const}$, $a \neq 0,\ -1$.

Then equation (4.2) becomes

$$\sum_{i,j}{}' A_{ij}u_{x_i}u_{x_j} = Cu^2/(a+1). \tag{5.1}$$

For exaple, a common solution of (1.2) and (5.1) is given by

$$u = \sum_{j=1}^{m} b_j \exp\Big(\sum_i a_{ij}x_i\Big), \tag{5.2}$$

where $a_{ij}, b_j (i = 1,\dots,n,\ j = 1,\dots,m)$ are constants satisfying

$$\begin{gathered}\sum_i a_{ij}B_i + (a+2)C/(a+1) = 0,\ \sum_{i,k}{}' A_{ik}a_{ij}a_{kj} = C/(a+1)\\ \sum_{i,l}{}' A_{il}(a_{ik}a_{ij}+a_{lj}a_{lk}) = 2C/(a+1),\ (j,k = 1,\dots,m, j<k)\end{gathered} \tag{5.3}$$

The function G is a solution of the equation

$$G''(t) + L(a+1)a^{-2}t^{-2(k+a+2)/a}G(t)^{-2k-3} = 0. \tag{5.4}$$

A particular solution of (5.4) is $G(t) = \Big(-La^2(a+1)^{-2}\Big)^{1/(2k+4)} t^{1/a}$ (see [**26**]), which means that z, given by

$$Q(z)/R(z) = -La^2(a+1)^{-2}u^{(2k+4)(a+1)/a}v^{-(2k+4)/a},$$

is a solution of (2.1).

The case $a=-2,\ a=-k-2,\ a=-(k+2)/(k+1)$ are of the interest, since (5.4) is in those cases integrable. Function G, in these cases is given by

$$\begin{cases} \int \Big(c + G^2/4t - Lt^{k+1}G^{-2k-2}/4(k+1)\Big)^{1/2} d(t^{-1/2}G) = \log\, t + d,\ a=-2 \\ \int \Big(c - LG^{-2k-2}/(k+2)^2\Big)^{1/2} dG = t + d,\ a = -k-2, \\ \int \Big(c - L(G/t)^{-2k-2}/(k+2)^2\Big)^{-1/2} d(G/t) = d - 1/t,\ a = -(k+2)/(k+1), \end{cases} \tag{5.5}$$

respectively, (c, d are arbitray constants).

Specially, for $c=0$, solutions of (2.1) be expressed in closed form.

We note, that for $B_i = 0\,(i=1,\dots,n)$, $C \neq 0$, (5.3) will be satisfied only for $a=-2$.

6. Nonlinear Klein-Gordon's equation. Now, we shall apply the previous results to the nonlinear Klein-Gordon's equation:

$$\Box^2\Phi + M^2\Phi + \lambda\Phi^{-2k-3} = 0 \qquad (M, \lambda, k = \text{const.}), \tag{6.1.}$$

where $\Box^2 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$.

This equation has the solution

$$\Phi = uG(1/u^2), \tag{6.2}$$

where G is determined by (5.5), case $a=-2$. Specially, for $c=0$, we have

$$\Phi = u\Big(\pm(\lambda/(k+1)dM^2)^{1/2}(du^{-2k-4}-1)^{1/(k+2)}\Big).$$

Function u is a common particular solution of the linear Klein-Gordon's equation

$$\Box^2 u + M^2 u = 0,$$

and the first order equation

$$u_t^2 - u_{x_1}^2 - u_{x_2}^2 - u_{x_3}^2 = M^2u^2.$$

Particularly, for $k=-3$ we obtain very important equation (see, for example, [**15-17**])

$$\Box^2\Phi + M^2\Phi + \lambda\Phi^{-3} = 0.$$

This equation has the sollution (6.2), where G(t) is determined by

$$\int (c + G^2/4t + \lambda t^{-2}G^4/8M^2)^{-1/2} d(t^{-1/2}G) = d + \log t,$$

(c, d are arbitray constants). Thus, in this case G is expressed through elliptic functions. For $c=0$ we obtain the result from [**15-17, 21**]

$$\Phi = u\Big(\pm(-\lambda/2M^2 d)^{1/2}(du^2-1)^{-1}\Big).$$

7. Ordinary differential equations. In this chapter we shall consider the case $n=1$, i.e. ordinary differential equations

(i) As a consequence of the theorem 1 we get the following:

Theorem 2. *Equation*

$$y'' + f(y', y, wh(u,v), x) = 0 \tag{7.1}$$

has the solution $y = F(u,v)$, *where* u, v *are two linearly independent patricular solutions of the linear equation:*

$$u'' + A(x)u' + B(x)u = 0, \tag{7.2}$$

if and only if it has the form

$$y'' + P(y)y'^2 + A(x)y' + B(x)Q(y) + C(x)R(y) = 0. \tag{7.3}$$

P, Q, R *satisfy* (2.2), C *is given by*

$$C(x) = Kw^2h(u,v)^2, \tag{7.4}$$

where K *is a constant,* w *is the Wronskian for* u *and* v, h *is homogenous function of order* k *in two variables.*

Function F *is given* (2.4.), *where* (2.5), (2.6) *hold.*

(ii) By the transformation $z(t) = \Big(Q(y)/R(y)\Big)^{1/2k+4}/u$, $t = v/u$, equation (7.3) reduces to the eqution of the form

$$z''(t) + a(t)z(t)^{-2k-3} = 0. \tag{7.5}$$

In the other words, equation (7.5) has the role of "canonical eqution" for the considered class of problems. Namely, all equations of the form (7.1) having solution $y = F(u,v)$ can be obtained from (7.5) by the substitution of the form $z = \xi(x)\zeta(y)$, $t = \eta(x)$.

(iii) Specially, for $h(u,v)=1$, we get the Herbst's equation (see, for example [**5, 9, 10, 12**]).

(iv) Equation (7.3), where $P(y)=(a-1)/y$, $Q(y)=y/a$, $R(y)=y^{-(2k+4)a+1}/a$ ($a=\text{const}$) and h has the form (3.3) is considered in papers [**18-21, 24**] in some special cases.

(v) Let $B(x)=0$, $P=0$, $Q=y$, $R=y^{-2k-3}$, then (7.3) becomes generalized Emden's equation (see, for example [**25-27**]). Taking, for example, $h(u,v)^2=(C_1u^2+C_2uv+C_3v^2)^k$ and using the result from section 3, point (iv), item 1, we obtain the conditions for the integrability of Emden's equation from [**25-27**].

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