PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 34 (48), 1983, pp. 109–116

LINEAR COMBINATIONS OF REGULAR FUNCTIONS WITH NEGATIVE COEFFICIENTS

G. Lakshma Reddy and K. S. Padmanabhan

Abstract. Let $f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+k} z^{n+k}$, $k \geq p \geq 1$ with $a_p > 0$, $a_{n+k} \geq 0$ be regular in $E = \{z : |z| < 1\}$ and $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$, $z \in E$ where $\lambda \geq 0$. The radius of p-valent starlikeness of order $\alpha, 0 \leq \alpha < 1$, of F as f varies over a certain subclass of p-valent regular functions in E is determined, and the mapping properties of F in certain other situations also are discussed.

1. Introduction. Let $E = \{z : | z | < 1\}$ be the unit disc in $\mathbf{C}$ and $H = \{w : w$ is regular in E such that $w(0) = 0$, $| w(z) | < 1$, $z \in E\}$. Let $P(A, B)$ denote the class of functions regular in E which are of the form $(1 + Aw(z))/(1 + Bw(z))$, $-1 \leq A < B \leq 1$, $w \in H$. Let T_p be the class of functions $f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+k} z^{n+k}$, $k \geq p \geq 1$, $a_p > 0$ and $a_{n+k} \geq 0$, regular in the unit disc E. Let $S_p^*(A, B) = \left\{ f \in T : \frac{1}{p} \frac{zf'(z)}{f(z)} \in P(A, B) \right\}$ and $K_p(A, B) = \left\{ f \in T_p : \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \in P(A, B) \right\}$. We note that $S_p^*(A, B)$ and $K_p(A, B)$ are subclasses of T_p consisting of p-valently starlike and p-valently convex functions respectively. $f \in S_p^*(A, B)$ implies that, $Re\{zf'(z)/f(z)\} > 0$ for $z \in E$. Further if $f \in S_p^*(A, B)$ and $z = re^{i\theta}$, $r < 1$, $\frac{1}{2\pi} \int_0^{2\pi} Re \frac{zf'(z)}{f(z)} d\theta = \frac{p}{2\pi} \int_0^{2\pi} Re \frac{1 + Aw(z)}{1 + Bw(z)} d\theta = p$, since $Re \frac{1+Aw(z)}{1+Bw(z)}$ is a harmonic function in E with $w(0) = 0$. This argument shows the p-valence of f in $S_p^*(A, B)$. Similarly $f \in K_p(A, B)$ is p-valently convex in E. Define $P^*(A, B) = \{f \in T_1 : f'(z) \in P(A, B), a_1 = 1\}$. In this paper we consider the function F in E defined by $F(z) = (1 - \lambda)f(z) + \lambda z f'(z)$, $\lambda \geq 0$ and study some mapping properties of F, as f varies over the classes $S_p^*(A, B)$, $K_p(B, A)$ and $P^*(A, B)$. We also consider the class of function $f \in T_p$ for which $f^{(p-1)}$ is univalent and discuss a mapping property of F. It is interesting to note that the necessary condition for univalence of $f^{(p-1)}$ for $f \in T_p$, turns out to be a necessary and sufficient condition for $f^{(p-1)}$ to be starlike univalent in E.

2. Coefficient inequalities and theorems on radius of starlikeness, convexity and close-to-convexity. We use the following notations for the sake of brevity

$$n+k=m,\ \ m(B+1)-p(A+1)=C_m,\ \ p(B-A)=D \ \text{ and } \ \sum_{m=k+1}^{\infty}=\Sigma.$$

We begin by proving the following

LEMMA 1. *Let $f \in T_p$. Then $f \in S_p^*(A,B)$ if and only if*

$$\Sigma C_m a_m \le D a_p. \tag{1}$$

Proof. Suppose $f \in S_p^*(A,B)$. Then $\frac{zf'(z)}{f(z)} = p\frac{1+Aw(z)}{1+Bw(z)}$, $-1 \le A < B \le 1$, $w \in H$.

That is,

$$w(z)=\frac{p-zf'(z)/f(z)}{Bzf'(z)/f(z)-Ap},\ w(0)=0 \ \text{ and}$$
$$\mid w(x)\mid=\left|\frac{zf'(z)-pf(z)}{Bzf'(z)-Apf(z)}\right|=\left|\frac{\Sigma(m-p)a_m z^m}{Da_p z^p-\Sigma(Bm-Ap)a_m z^m}\right|<1.$$

Thus

$$\mathrm{Re}\left\{\frac{\Sigma(m-p)a_m z^m}{Da_p z^p \Sigma(Bm-Ap)a_m z^m}\right\}<1. \tag{2}$$

Take $z=r$ with $0<r<1$. Then, for sufficiently small r, the denominator of the left hand member of (2) is positive and so it is positive for all r $0<r<1$, since $w(z)$ is regular for $\mid z \mid < 1$. Then (2) gives

$$\Sigma(m-p)a_m r^m < Da_p r^p - \Sigma(Bm-Ap)a_m r^m, \ \text{ that is,}$$
$$\Sigma[m(B+1)-p(A+1)]a_m r^m < Da_p r^p, \ \text{ that is, } \ \Sigma C_m a_m r^m < D a_p r^p,$$

and (1) follows on letting $r \to 1$.

Conversely, for $\mid z \mid = r$, $0<r<1$, and since $r^m < r^p$, by (1) we have $\Sigma C_m a_m r^m < r^p \Sigma C_m a_m < D a_p r^p$. Using this inequality we have

$$\mid \Sigma(m-p)a_m z^m \mid \le \Sigma(m-p)a_m r^m < Da_p r^p - \Sigma(Bm-Ap)a_m r^m$$
$$\le \mid Da_p z^p - \Sigma(Bm-Ap)a_m z^m \mid .$$

This proves that $zf'(z)/f(z)$ is on the form $p\frac{1+Aw(z)}{1+Bw(z)}$ with $w \in H$. Therefore $f \in S_p^*(A,B)$ and the proof is complete.

COROLLARY 1. $f \in T_1$. *Then* $f \in S_1^*(2\alpha - 1, 1)$ *if and only if* $\Sigma(m-\alpha)a_m \leq (1-\alpha)a_1$.

Remark. Note that $S_1^*(2\alpha-1,\ 1) = S^*(\alpha)$, the class of univalent starlike functions of order α. This corollary reduces to Theorem 1 in [**2**].

THEOREM 1. *Let* $f \in S_p^*(A,B)$ *and* $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$, $\lambda \geq 0$, $z \in E$. *Then* F *is p-valently starlike of order* α, $0 \leq \alpha < 1$, *for*

$$\mid z \mid < r_1 = \inf_m \left[\frac{(p-\alpha)(1+(p-1))}{(m-\alpha)(1+(m-1))} \cdot \frac{C_m}{D} \right]^{1/(m-p)} \quad \textit{and the bound is sharp.}$$

Proof. Since $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$, $\lambda \geq 0$

$$= (1+(p-1)\lambda)a_p z^p - \Sigma(1+(m-1)\lambda)a_m z^m,$$

$$\frac{zF'(z)}{F(z)} = \frac{p(1+(p-1)\lambda)a_p z^p - \Sigma m(1+(m-1)\lambda)a_m z^m}{(1+(p-1)\lambda)a_p z^p - \Sigma(1+(m-1)\lambda)a_m z^m}.$$

It suffices to show that $\mid zF'(z)/F(z) - p \mid \leq p - \alpha$ for $\mid z \mid < r_1$. Now

$$\begin{aligned} \left| \frac{zF'(z)}{F(z)} - p \right| &= \left| \frac{-\Sigma(m-p)(1+(m-1)\lambda)a_m z^m}{(1+(p-1)\lambda)a_p z^p - \Sigma(1+(m-1)\lambda)a_m z^m} \right| \\ &\leq \frac{\Sigma(m-p)(1+(m-1)\lambda)a_m \mid z \mid^{m-p}}{\mid (1+(p-1)\lambda)a_p - \Sigma(1+(m-1)\lambda)a_m \mid z \mid^{m-p} \mid}. \end{aligned} \tag{3}$$

Consider the values of z for which $\mid z \mid < r_1$, so that

$$\mid z \mid^{m-p} \leq \frac{(p-\alpha)(1+(p-1)\lambda)}{(m-\alpha)(1+(m-1)\lambda)} \cdot \frac{C_m}{D} \text{ holds. Then}$$

$$\begin{aligned} \Sigma \frac{1+(m-1)\lambda}{1+(p-1)\lambda} \cdot a_m \mid z \mid^{m-p} &\leq \Sigma \frac{(p-\alpha)}{(m-\alpha)} \cdot \frac{C_m}{D} a_m \\ &\leq \Sigma \frac{C_m}{D} a_m < a_p, \text{ which is true by (1).} \end{aligned}$$

Thus, the expression within the modulus sign in the denominator of the right hand side of (3), for the considered values of z is positive and so we have

$$\left| \frac{zF'(z)}{(Fz)} - p \right| \leq \frac{\Sigma(m-p)(1+(m-1)\lambda)a_m \mid z \mid^{m-p}}{(1+(p-1)\lambda)a_p - \Sigma(1+(m-1)\lambda)a_p \mid z \mid^{m-p}} \leq p - \alpha \text{ if}$$
$$\Sigma(m-\alpha)(1+(m-1)\lambda)a_m \mid z \mid^{m-p} \leq (p-\alpha)(1+(p-1)\lambda)a_p,$$

that is, if

$$\Sigma \frac{(m-\alpha)(1+(m-1)\lambda)a_m \mid z \mid^{m-p}}{(p-\alpha)(1+(p-1)\lambda)a_p} \leq 1. \tag{4}$$

By Lemma 1 we have $f \in S_p^*(A,B)$ if and only if $\Sigma \frac{C_m a_m}{D a_p} \leq 1$. Hence (4) is true if

$$\frac{(m-\alpha)(1+(m-1)\lambda)}{(p-\alpha)(1+(p-1)\lambda)} \mid z \mid^{m-p} \leq \frac{C_m}{D},$$

that is, if

$$\mid z \mid \leq \left[\frac{(p-\alpha)(1+(p-1)\lambda)}{(m-\alpha)(1+(m-1)\lambda)} \cdot \frac{C_m}{D}\right]^{1/(m-p)}.$$

To see the p-valence of F in $\mid z \mid < r_1$, we observe that $zF'(z)/F(z)$ is regular in $\mid z \mid < r_1$ and hence $\mathrm{Re}(zF'(z)/F(z))$ is harmonic in that disc. For $r < r_1$ and $z = re^{i\theta}$, $\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\frac{zF'(z)}{F(z)} d\theta = p$, showing that F is p-valent in $\mid z \mid < r_1$.

Hence the proof follows. The extremal function is given by $f(z) = a_p z^p - (Da_p/C_m)z^m$.

Remark. We obtain Theorem 2 in [**1**] as a particular case.

Corollary 2. *If $f \in S_p^*(A,B)$, then f is p-valently starlike of order α, $0 \leq \alpha < 1$, in*

$$\mid z \mid < \inf_m \left[\frac{(p-\alpha)}{(m-\alpha)} \cdot \frac{C_m}{D}\right]^{1/(m-p)}, \ m = k+1, k+2, \ldots .$$

Proof. Put $\lambda = 0$ in Theorem 1.

Corollary 3. *If $f \in S_p^*(A,B)$, then f is p-valently convex of order β, $0 \leq \beta < 1$, in*

$$\mid z \mid < \inf_m \left[\frac{p}{m} \cdot \frac{(p-\beta)}{(m-\beta)} \cdot \frac{C_m}{D}\right]^{1/(m-p)}, \ m = k+1, \ldots .$$

Proof. Put $\lambda = 1$ in Theorem 1 and note that $zf'(z)$ is starlike of order β if and only if $f(z)$ is convex of order β.

Remark. For $k = p = 1,\ A = 2\alpha - 1,\ B = 1,\ \beta = 0$, Corollary 3 reduces to Theorem 8 in [**2**].

Corollary 4. *If $f \in S_p^*(A,B)$ and $F(z) = \frac{(z^c f(z))'}{1+c}$, $c = 1, 2, \ldots, z \in E$, then F is p-valently starlike of order α, $0 \leq \alpha < 1$, in*

$$\mid z \mid \leq \inf_m \left[\frac{(p-\alpha)(p+c)}{(m-\alpha)(m+p)} \cdot \frac{C_m}{D}\right]^{1/(m-p)}.$$

Proof. Put $\lambda = 1/(c+1)$ in Theorem 1.

Theorem 2. *Let $f \in K_p(A,B)$ and $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$, $\lambda \geq 0$, $z \in E$. Then $\frac{(F'/d')}{(1+(p-1)\lambda)} \in P(A_1, B)$, where $A_1 = \frac{A\lambda p + (1-\lambda)B}{1+(p-1)\lambda}$, if $\lambda < \frac{1+B}{1+B-p(A+1)}$ and*

$p < \frac{1+B}{1+A}$. In particular F is p-valently close-to-convex in E if $\lambda < \frac{1+B}{1+B-p(A+1)}$ and $p < \frac{1+B}{1+A}$. Also F is p-valently convex of order α, $0 \leq \alpha < 1$, in $\mid z \mid < r_1$. This bound is sharp.

Proof. Since $f \in K_p(A, B)$. We have

$$\frac{F'(z)}{f'(z)} = (1-\lambda) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 - \lambda + \lambda p \frac{1 + Aw(z)}{1 + Bw(z)}$$
$$= \frac{(1 + (p-1)\lambda) + (A\lambda p + B(1-\lambda))w(z)}{1 + Bw(z)}, \text{ where } w \in H.$$

Therefore

$$\frac{1}{(1 + (p-1)\lambda)} \frac{F'(z)}{f'(z)} = \frac{1 + A_1 w(z)}{1 + Bw(z)} \text{ where } A_1 = \frac{A\lambda + p + B(1-\lambda)}{1 + \lambda(p-1)}.$$

Evidently $A_1 < B$ since $A < B$ and $A_1 > -1$, provided $\lambda < \frac{B+1}{1+B-p(A+1)}$ and $p < \frac{1+B}{1+A}$. Therefore, $\frac{(F'/f')}{1+(p-1)\lambda} \cdot \in P(A_1, B)$ with $-1 < A_1 < B \leq 1$. In particular $\operatorname{Re} F'/f' > 0$. Further since f is p-valently convex, it follows that F is p-valent by a theorem of Umezewa [**3**, Theorem 1]. It follows that F, if p-valently close-to-convex in E if $\lambda < \frac{B+1}{B+1-p(A+1)}$ and $p < \frac{B+1}{A+1}$. We now prove that $F(z)$ is convex of order α, $0 \leq \alpha < 1$, in $\mid z \mid < r_1$ where r_1 is as given in Theorem 1. $zF'(z) = (1-\lambda)zf'(z) + \lambda z(zf'(z))'$ for $z \in E$. $f \in K_p(A, B)$ if and only if $zf' \in S_p^*(A, B)$. Applying Theorem 1 to zf' we conclude that $zF'(z)$ is p-valently starlike of order α for $\mid z \mid < r_1$. So, F is p-valently convex of order α for $\mid z \mid < r_1$. The extremal function is given by $f(z) = a_p z^p - (Dpa_p/mC_m)z^m$.

Remark. We get Theorem 3 in [**1**] as a special case.

Lemma 2. *Let $f \in T_1$, $a_1 = 1$. Then $f \in P^*(A, B)$ if and only if*

$$\Sigma m(B+1)a_m \leq B - A. \tag{5}$$

Proof. Proof of Lemma 2 is similar to the proof of Lemma 1 and is hence omitted.

Theorem 3. *Let $f \in P^*(A, B)$ and $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$ for $z \in E$, $\lambda \geq 0$. Then*

$$\operatorname{Re} F'(z) > \beta,\ 0 \leq \beta < 1, \quad \textit{for} \quad \mid z \mid < r_2 = \inf_m \left[\frac{(1-\beta)}{1 + (m-1)\lambda} \frac{B+1}{B-A}\right]^{1/(m-1)}$$

and the bound is sharp.

Proof. It suffices to show that $\mid F'(z) - 1 \mid \leq 1 - \beta$ for $\mid z \mid < r_2$. Since $f \in P^*(A, B)$, using Lemma 2, we see that (5) holds. Since $F'(z) = 1 - \Sigma m(1 +$

$(m-1)\lambda)a_m z^{m-1}$, using (5), we see that $\mid F'(z)-1 \mid \leq \Sigma m(1+(m-1)\lambda)a_m \mid z \mid^{m-1} \leq 1-\beta$ provided

$$\frac{m(1+(m-1)\lambda)}{1-\beta} \mid z \mid^{m-1} \leq \frac{m(B+1)}{B-A}. \tag{6}$$

Now (6) holds if $\mid z \mid \leq \left[\frac{(1-\beta)}{1+(m-1)\lambda} \cdot \frac{B+1}{B-A}\right]^{1/(m-1)}$, and the proof follows.

The extremal function is $f(z) = z - \dfrac{B-A}{m(B+1)} z^m$.

Remark. Theorem 4 in [**1**] arises as a special case.

LEMMA 3. *Let $f \in T_p$ and let $f^{(p-1)}$ be univalent in E. Then*

$$\Sigma[m(m-1)\dots(m-p+1)]a_m \leq p!\, a_p. \tag{7}$$

Proof. Suppose $\Sigma[m(m-1)\dots(m-p+1)]a_m = p!a_p+\varepsilon,\ \varepsilon>0$. Then there exists a positive integer $N > p$ such that

$$\sum_{m=k+1}^{N} [m(m-1)\dots(m-p+1)]a_m > p!a_p + \varepsilon/2. \tag{8}$$

Let $\left(\frac{p!a_p}{p!a_p+\varepsilon/2}\right)^{1/(N-p)} < z < 1$, so that $z^{N-p} > \frac{p!a_p}{p!a_p+\varepsilon/2}$. Also $z^{m-p} > z^{N-p}$ for $m < N$, since $z < 1$. Using these two inequalities and (8), we have

$$\begin{aligned}
f^{(p)}(z) &= p!a_p - \Sigma[m(m-1)\dots(m-p+1)]a_m z^{m-p} \\
&\leq p!a_p - \sum_{m=k+1}^{N} [m(m-1)\dots(m-p+1)]a_m z^{m-p} \\
&\leq p!a_p - z^{N-p} \sum_{m=k+1}^{N} [m(m-1)\dots(m-p+1)]a_m \\
&< p!a_p - (p!a_p+\varepsilon/2)z^{N-p} < 0,
\end{aligned}$$

and $f^{(p)}(0) = p!a_p > 0$. Therefore, there exists a point z_0 with $0 < z_0 < \left(\frac{p!a_p}{p!a_p+\varepsilon/2}\right)^{1/(N-p)} < 1$ such that $f^{(p)}(z_0) = 0$. This contradicts the univalence of $f^{(p-1)}(z)$ for $z \in E$. Hence the lemma.

Remarks 1. Putting $p=1,\ a_p=1$, this lemma reduces to Theorem 3 in [**2**].

2. Applying Corollary 1 with $\alpha = 0$ to $f^{(p-1)}(z)$ which belongs to the class T_1, we see that condition (7) is necessary and sufficient for $f^{(p-1)}$ to be starlike univalent.

THEOREM 4. *Let* $f \in T_p$ *and let* $f^{(p-1)}$ *be univalent in* E. *Suppose* $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$, $\lambda \geq 0$, $z \in E$. *Then* $\mathrm{Re}\{zF'(z)/F(z)\} < \alpha$, $0 \leq \alpha < 1$, *for*

$$\mid z \mid < r_3 = \inf_m \left[\frac{(p-\alpha)[1+(p-1)\lambda]}{(m-\alpha)[1+(m-1)\lambda]} \frac{m(m-1)\dots(m-p+1)}{p!} \right]^{1/(m-p)}$$

and the bound is sharp.

Proof. Let $F(z) = (1-\lambda)f(z) + \lambda z f'(z)$. Then inequality (3) holds. Now, since $f \in T_p$, by Lemma 3 inequality (7) holds. Consider the values of z for which $\mid z \mid < r_3$ so that $\mid z \mid^{m-p} \leq \frac{(p-\alpha)(1+(p-1)\alpha)}{(m-\alpha)(1+(m-1)\lambda)} \cdot \frac{m(m-1)\dots(m-p+1)}{p!}$.

Then

$$\begin{aligned} \Sigma \frac{1+(m-1)\lambda}{1+(p-1)\lambda} a_m \mid z \mid^{m-p} &\leq \Sigma \frac{(p-\alpha)}{(m-\alpha)} \frac{m(m-1)\dots(m-p+1)}{p!} a_m \\ &< \Sigma \frac{m(m-1)\dots(m-p+1)}{p!} a_m \\ &< a_p, \quad \text{by}(7). \end{aligned}$$

Therefore as in the proof of Theorem 1, we can write $\mid zF'(z)/F(z) - p \mid \leq p - \alpha$ provided (4) holds. Using (7), we see inequality (4) also holds provided

$$\frac{(m-\alpha)(1+(m-1)\lambda)}{(p-\alpha)(1+(p-1)\lambda)} \mid z \mid^{m-p} \leq \frac{m(m-1)\dots(m-p+1)}{p!}, \text{ that is, if}$$

$$\mid z \mid \leq \left[\frac{(p-\alpha)(1+(p-1)\lambda)}{(m-\alpha)(1+(m-1)\lambda)} \cdot \frac{m(m-1)\dots(m-p+1)}{p!} \right]^{1/(m-p)}.$$

The proof follows. The extremal function is given by

$$f(z) = a_p z^p - \frac{p! a_p}{m(m-1)\dots(m-p+1)} z^m.$$

It is easy to verify the univalence of $f^{(p-1)}(z)$ in E.

REFERENCES

[1] S. S. Bhoosnurmath, S. R. Swamy, *Analytic functions with negative coefficients*, Indian J. Pure Appl. Math., **12(6)** (1981), 738–742.

[2] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.

[3] T. Umezawa, *Multivalently close-to-convex functions*, Proc. Amer. Math. Soc. **8** (1957), 869–874.

Ramanujan Institute for Advanced Study in Mathematics
University of Madras,
Madras, 600 005, India

(Received 10 04 1982)