PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 34 (48), 1983, pp. 121–124

# ON $\Sigma_1^0$-EXTENSIONS OF $\boldsymbol{\omega}$

**Žarko Mijajlović**

We shall consider $\Sigma_1^0$-extensions of the standard model of arithmetic by models of Peano arithmetic. Even if the notation we shall use is standard in the study of models of Peano arithmetic, we recall some of it.

Peano arithmetic is denoted by $P$, and its language by $L_p$. By $\mathbf{M}$, $\mathbf{N}$, etc. we denote models of $L_p$, or simple expansions of models of this language, and $M, N, \dots$ stand for their domains respectively. The standard model of natural numbers is denoted by $\boldsymbol{\omega}$, thus $\boldsymbol{\omega} = (\omega, +, \cdot, ', 0)$. The sequence $a_0, \dots, a_n \in M$ will be abbreviated by $\vec{a} \in M$. If $a \in M$, then $\mathbf{a}$ denotes the name of $a$.

The symbols $\subseteq_e, \prec_e, \subseteq_c, \prec_c$ have the usual model-theoretic meaning, namely, they denote respectively the relations of: end-extension, elementary end-extension, cofinal extension, and elementary cofinal extension. If $\Phi$ is some of the usual arithmetical hierarchies, e. g. $\Sigma_n^0, \Pi_n^0$, then the symbol $\prec_\Phi$ is used to denote the restriction of the relation $\prec$ to the formulas of $\Phi$.

By Sat $_{\Sigma_k^0}(x, y_1, \dots, y_n)$ we denote a formula of $L_p$ which represents the satisfaction relation for $\Sigma_k^o$ formulas of $P$ with free variables $y_1, \dots, y_n$. If $\mathbf{M}$ is a model of $P$, then SSy($\mathbf{M}$) denotes the standard system of $\mathbf{M}$, i.e. the collection of all sets of the form $\{n \in \omega : \mathbf{M} \models \varphi \mathbf{n} \vec{b}\}$ for some $\vec{b} \in M$ and a formula $\varphi$ of $L_p$. Observe that SSy($\boldsymbol{\omega}$) is exactly the set of all definable subsets of $\boldsymbol{\omega}$. By Th($\mathbf{M}$) we denote the set of all sentences of $L_p$ true in $\mathbf{M}$.

We are now able to state precisely the question we are interested in. Assume $\boldsymbol{\omega} \prec_{\Sigma_1^0} \mathbf{M}$, where $\mathbf{M}$ is a model of $P$. By a classical compactness argument there is a model $\mathbf{N}$ such that $\omega \subseteq \mathbf{M} \subseteq \mathbf{N}$, and $\boldsymbol{\omega} \prec \mathbf{N}$. Using Gaifman's splitting theorem, cf. [**4 b** Theorem 2.5]; we can find $\mathbf{K}$ such that $\mathbf{M} \prec_c \mathbf{K} \subseteq_e \mathbf{N}$. Thus, we proved the following sandwich-type fact:

Let $\omega \prec_{\Sigma_1^0} \mathbf{M}, \mathbf{M} \models P$. Then there are $\mathbf{K}$, $\mathbf{N}$ so that $\boldsymbol{\omega} \subseteq \mathbf{M} \prec_c \mathbf{K} \subseteq_e \mathbf{N}$, and Th($\mathbf{N}$)=Th($\boldsymbol{\omega}$).

*AMS Subject Classification* (1980): Primary 03 H 15, 03 H 20.

Now we turn to the natural question raised by this proposition: which $\Sigma_1^0$ exstansions of $\boldsymbol{\omega}$ have end-extension elementary equivalent to $\boldsymbol{\omega}$? We shall answer this question completely for countable models.

THEOREM 1. *Let* $\mathbf{N}$ *be a countable model of* $P$ *such that* $\omega \prec_{\Sigma_1^0} \mathbf{N}$. *Then the following are equivalent:*

1° *There is a model* $\mathbf{M}$ *of* $L_p$ *such that* $\mathbf{N} \subseteq_e \mathbf{M}$, *and* $\boldsymbol{\omega} \prec \Sigma_1^0\mathbf{M}$

2° $SSy(\boldsymbol{\omega}) \subseteq SSy(\mathbf{N})$.

3° *For each* $n \in \omega$ *the code set of* $Th(\boldsymbol{\omega}) \cap \Sigma_n^0$ *belongs to* $SSy(\mathbf{N})$.

*Proof*. (1° $\to$ 2°) If $\varphi x$ defines $A \subseteq \omega$, then $A = \{n \in \omega : \boldsymbol{\omega} \models \varphi\mathbf{n}\} = \{n \in \omega : \mathbf{M} \models \varphi\mathbf{n}\}$, hence $A \in SSy(\mathbf{M})$. As $\mathbf{N} \subseteq_e \mathbf{M}$, it follows that $SSy(\mathbf{N}) = SSy(\mathbf{M})$, i.e. $A \in SSy(\mathbf{N})$.

(2° $\to$ 3°) Assume $SSy(\boldsymbol{\omega}) \subseteq SSy(\mathbf{N})$. The code set $A$ of $Th(\boldsymbol{\omega}) \cap \Sigma_n^0$ is definable in $\boldsymbol{\omega}$ for each $n$ as $A = \{\ulcorner\varphi\urcorner : \varphi \in \Sigma_n^0,\ \boldsymbol{\omega} \models \mathrm{Sat}_{\Sigma_n^0}(\ulcorner\varphi\urcorner)\}$, where $\ulcorner\varphi\urcorner$ denotes the code number of the formula $\varphi$. Hence $A \in SSy(\mathbf{N})$.

(3° $\to$ 1°) This part follows immediately by the following theorem of D. Jensen and A. Ehrenfeucht (cf. [**2**, Theorem 3]):

If $\mathbf{M}$ is a countable nonstandard model of $P$ and $T$ is a complete extension of $P$, then there is an end-extension $\mathbf{N}$ of $\mathbf{M}$ with $\mathrm{Th}(\mathbf{N})=T$ iff both of the following holds:

(i) $\mathrm{Th}(\mathbf{M})\cap\Sigma_1^0 \subseteq T$.

(ii) For each $n \in \omega$ the code set of $T \cap \Sigma_n^0$ belongs to $\mathrm{SSy}(\mathbf{M})$.

Originally, it was assumed in (ii) that $T \cap B_n$ belongs to $\mathrm{SSy}(\mathbf{M})$, where $B_n$ denotes the set of all Boolean combinations of $\Sigma_n^0$ formulas. However, this is not a stronger assumption as $\mathrm{SSy}(\mathbf{M})$ is a field of sets.

In [**5**] a criterion is given for a model $\mathbf{M}$ having an end-extension regardless of the cardinality of $\mathbf{M}$. A variant of Theorem 1 in [**5**] is:

THEOREM 2. *Let* $\mathbf{M}$ *be a nonstandard model of* $P$ *and* $T$ *a complete extension of* $P$. *Then there is a model* $\mathbf{M}_1$ *of* $T$ *satisfying* $\mathbf{M} \subseteq_e \mathbf{M}_1$ *iff both of the following conditions hold:*

1° $\mathrm{Th}(\mathbf{M}) \cap \Sigma_1^0 \subseteq T$.

2° *There is a model* $\mathbf{M}_2$ *of* $T$ *such that* $\mathrm{SSy}(\mathbf{M}_2) = \mathrm{SSy}(\mathbf{M})$.

Thus, taking $T = \mathrm{Th}(\boldsymbol{\omega})$ we have the following

COROLLARY 2.1. *Let* $\mathbf{N}$ *be a model of* $P$ *such that* $\boldsymbol{\omega} \prec_{\Sigma_1^0} \mathbf{N}$. *Then the following propositions are equivalent:*

1° *There is a model* $\mathbf{M}$ *such that* $\mathbf{N} \subseteq_e \mathbf{M}$ *and* $\boldsymbol{\omega} \prec \mathbf{M}$.

2° *There is a model* $\mathbf{M}$ *of* $\mathrm{Th}(\boldsymbol{\omega})$ *such that* $\mathrm{SSy}(\mathbf{M}) = \mathrm{SSy}(\mathbf{N})$.

Some further improvements can be made. For example, using the Arithmetizated Completeness Theorem we obtain at once for any model $\mathbf{M}$ of $P$ such that

$\boldsymbol{\omega} \prec_{\Sigma_1^0} \mathbf{M}$ and $\mathrm{Th}(\boldsymbol{\omega}) \in \mathrm{SSy}(\mathbf{M})$, that there is a (recursively saturated) model $\mathbf{N}$ which satisfies $\mathbf{M} \subseteq_e \mathbf{N}$ and $\mathbf{N} \models \mathrm{Th}(\boldsymbol{\omega})$. For related details one may consult [**4a**, p. 860], and also Lemma 2 in [**5**].

Now, we shall exhibit a countable $\Sigma_1^0$-elementary extension of $\boldsymbol{\omega}$ which has no end-extension elementary equivalent to $\boldsymbol{\omega}$. This result shows that Theorem 1 is not vacuous.

THEOREM 3. *There is a model* $\mathbf{M}$ *of* $P$ *such that* $\boldsymbol{\omega} \prec_{\Sigma_1^0} \mathbf{M}$ *and* $\mathrm{SSy}(\boldsymbol{\omega}) - \mathrm{SSy}(\mathbf{M}) \neq \varnothing$.

*Proof*. Let $T = P + \mathrm{Th}(\boldsymbol{\omega}) \cap \Sigma_2^0$. Then the code set of $T$ is $\Sigma_2^0$ as

$$\mathrm{Th}(\boldsymbol{\omega}) \cap \Sigma_2^0 = \{\varphi : \varphi \in \Sigma_2^0, \quad \boldsymbol{\omega} \models \mathrm{Sat}_{\Sigma_2^0}(\ulcorner\varphi\urcorner).$$

Now, define $A = \{n \in \omega \mid \boldsymbol{\omega} \models \mathrm{Sat}_{\Sigma_2^0}(\mathbf{m}, \mathbf{m})\}$. Thus, $A$ belongs to $\Sigma_2^0$, i.e. $A \in \mathrm{SSy}(\boldsymbol{\omega})$. If $A$ would be $\Pi_2^0$, then $A^c$ would be $\Sigma_2^0$, so for some $e \in \omega$ $A^c = \{m \in \omega : \boldsymbol{\omega} \models \mathrm{Sat}_{\Sigma_2^0}(\mathbf{e}, \mathbf{m})\}$. Hence, $e \in A$ iff $\boldsymbol{\omega} \models \mathrm{Sat}_{\Sigma_2^0}(\mathbf{e}, \mathbf{e})$ iff $e \in A^c$, a contradiction. Therefore $A$ is $\Sigma_2^0 - \Pi_2^0$. As $\mathrm{Th}(\boldsymbol{\omega} \cap \Sigma_2^0 \subseteq T$, we see that $A$ is recursively enumerable in $T$. However, $A$ is not recursive relative to $T$ as $A$ is not representable in $T$: if there were $\psi$ of $L_p$ such that $n \in A$ implies $T \vdash \psi\mathbf{n}$, and $n \notin A$ implies $T \vdash \rceil\psi\mathbf{n}$ then $A$ and $A^c$ would be recursively enumerable in $T$, and so, as $T$ is $\Sigma_2^0$, it would follow that $A^c$ is $\Sigma_2^0$, a contradiction.

Let $\varphi_0 x, \varphi_1 x, \dots$ be an enumeration of formulas of $L_p$ with $x$ as the only free variable. Define a sequence $T_n$, $n \in \omega$, of theories as follows:

Let $T_0 = T$. Assume $T_n$ has been defined. We distinguish two cases:

1° If $\{m \in \omega : T_n \vdash \varphi_n\mathbf{m}\}^c = \{m \in \omega : T_n \vdash \rceil\varphi_n\mathbf{m}\}$, then $T_{n+1} = T_n$.

2° Assume for some $m \in \omega$ not $T_n \vdash \varphi_n\mathbf{m}$ and not $T_n \vdash \rceil\varphi_n\mathbf{m}$.

If $m \in A$, then let $T_{n+1} = T_n \cup \{\rceil\varphi_n\mathbf{m}\}$

If $m \notin A$, then let $T_{n+1} = T_n \cup \{\varphi_n\mathbf{m}\}$.

Let $T' = \cup_{n\in\omega} T_n$ and let $\mathbf{M}$ be any prime model of $T'$. Then every $X \in \mathrm{SSy}(\mathbf{M})$ is definable in $\mathbf{M}$ without parameters. So if $A \in \mathrm{SSy}(\mathbf{M})$, then for some $n, A = \{m \in \omega : \mathbf{M} \models \varphi_n\mathbf{m}\}$. If $\varphi_n$ satisfied case 1°, then $A$ would be represented in $T$, a contradiction. Thus $\varphi_n$ satisfies case 2°; so for some $m \in \boldsymbol{\omega}$ not $T_n \vdash \varphi_n\mathbf{m}$, and not $T_n \vdash \rceil\varphi_n\mathbf{m}$. But then $A \in \mathrm{SSy}(\mathbf{M})$ contradicts the choice of $T_{n+1}$. Therefore $A \notin \mathrm{SSy}(\mathbf{M})$

COROLLARY 3.1. *There is a model* $\mathbf{M}$ *of* $P$ *such that* $\boldsymbol{\omega} \prec_{\Sigma_1^0} \mathbf{M}$ *and this model has no end-extension which is elementary equivalent to* $\boldsymbol{\omega}$.

## REFERENCES

[1] H. Gaifman, *Models and types of Peano's arithmetic*, Ann. Math. Logic **9** (1976), 223–306.

[2] D. Jensen, A. Ehrenfeucht, *Some problem in elementary arithmetic*, Fund. Math. **92** (1976), 223–245.

[3] D. Scott, "Algebra of sets binumerable in complete extensions of arithmetic", *Recursive function theory* (J. C. E. Dekker, Editor), AMS, Providence, R. I. 1962.

[4] C. Smorinski, a) "The incompleteness theorems", J. Barwise (ed.), *Handbook of Mathematical logic*, North Hollandd, Amsterdam 1977.

b) *Recursively saturated nonstandard models of arithmetic*, J. Symbolic Logic **46** (1981) 259–286.

[5] A. Wilkie, "On the theories of end-extension of models of arithmetic", *Set Theory and Hierarchy Theory V* (A. Lachlan, M. Srebrny and A. Zarach, Editors), Springer-Verlag, Heidelberg 1977.

Institut za matematiku
Prirodno-matematički fakultet
11000 Beograd
Jugoslavija

(Received 14 12 1982)