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THE $(\psi, \xi, \eta, \overline{g})$ STRUCTURE ON SUBSPACES OF THE SPACE WITH THE $\varphi(4, -2)$ STRUCTURE

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Abstract. Let a tensor field φ, $\varphi \neq 0$, $\varphi \neq 1$, of type (1,1) and of class C^∞ be given on M^n such that $\varphi^4 - \varphi^2 = 0$, and rank $\varphi = n - 1$. The structure $\Phi = 2\varphi - 1$ is an almost product structure. Φ induces on hypersurface K a Sato structure. In this paper it is proved that the structure Sato ψ induced by Φ on K^* is equal to the $\overline{\varphi}$.($\overline{\varphi}$ is the restriction of the structure φ on K^*).

Introduction. In [**1**] Yano, Houh and Chen consider the structure called a $\varphi(4, -2)$ structure, defined by a tensor field φ of type (1,1) satisfying $\varphi^4 - \varphi^2 = 0$ and they study the existence of this structure.

In this paper we study a $\varphi(4, -2)$ structure of rank $r = n - 1$ and the restriction of the structure φ on the hypersurface K. In **3.** we shall examine the relation between the almost product structure $\Phi = 2\varphi^2 - 1$ and $\varphi/_{K^*}$.

1. Preliminaries. Let $\mathcal{M}^n$ be an n-dimensional differentiable manifold of class C^∞, and let the $C^\infty(1,1)$ tensor fields f_1 and f_2 be given such that $f_1^2 = 1$, $f_1^2 = 0$. Then f_1 is an almost product structure, and f_2 is an almost tangent structure. Let a tensor field $\varphi, \varphi \neq 0$ and $\varphi \neq 1$, of type (1,1) and of class C^∞ be given on $\mathcal{M}^n$ such that $\varphi^4 - \varphi^2 = 0$ and rank $\varphi =$(rank $\varphi^2 + \dim \mathcal{M}^n)/2 = r$.

Let $\mathbf{l} = \varphi^2$, $\mathbf{m} = 1 - \varphi^2$, then $\varphi\mathbf{l} = \mathbf{l}\varphi = \varphi^3$, $\varphi\mathbf{m} = \mathbf{m}\varphi = \varphi - \varphi^3$, $\varphi^2\mathbf{l} = \mathbf{l}^2 = \mathbf{l}$, $\varphi^2\mathbf{m} = \mathbf{m}\varphi^2 = 0$.

Let $\Phi = \mathbf{l} - \mathbf{m} = 2\varphi^2 - 1$. Then it is clear that Φ defines on $\mathcal{M}^n$ an almost product structure if $\varphi^2 \neq 1$. Let L and M be the distributions corresponding to $\mathbf{l}$ and $\mathbf{m}$ respectively. We assume that $\varphi' = \varphi/L$ is not the identity operator of L. Then φ acts on L as an almost product structure operator and on M as an almost tangent structure operator. Moreover, $\dim M = 2(n - r)$ and $\dim L = 2r - n$. Such a structure φ is called a $\varphi(4, -2)$ structure of rank r.

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If the rank of φ is maximal, $r = n$, the $\varphi(4, -2)$-structure is an almost product structure and if the rank of φ is minimal, $2r = n$, the $\varphi(4, -2)$-structure is an almost tangent structure.

In [**1**] it has been proved that a necessary and sufficient condition for an n-dimensional manifold to admit a tensor field φ, $\varphi \neq 0$ and $\varphi \neq 1$ of type (1,1) defining a $\varphi(4, -2)$-structure is that the group of the tangent bundle of the manifold be reduced to the group $0(h) \times 0(2r - n - h) \times 0(n - r) \times 0(n - r)$ $h = \dim L_1$, L_1 being the subspace of L corresponding to the eigen value $+1$ of φ:

With respect to the adapted frame the tensors g_{ij} and φ^i_j have the components

$$g = \begin{bmatrix} E_h & 0 & 0 & 0 \\ 0 & E_{2r-n-h} & 0 & 0 \\ 0 & 0 & E_{n-r} & 0 \\ 0 & 0 & 0 & E_{n-r} \end{bmatrix} \quad \varphi = \begin{bmatrix} E_h & 0 & 0 & 0 \\ 0 & -E_{2r-n-h} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_{n-r} & 0 \end{bmatrix}$$

I. Sato [**2**] introduced and studied almost paracontact Riemannian manifold V with the structure (ψ, ξ, η, g) that is, an n-dimensional differentiable manifold with a tensor field ψ of type (1,1), a positive definite Riemannian metric g, a vector field ξ and a 1-form η satisfying

(1) $\psi^2 = I - \otimes\xi,\ \psi\xi = 0,\ \eta\psi = 0,\ \eta(\xi) = 1,$

(2) $\eta(X) = g(\xi, X),\ g(\psi X, \psi Y) = g(X, Y) - \eta(X)\eta(Y),\ X, Y \in \mathcal{X}(V)$

where I is the identity and $\mathcal{X}(V)$ denotes the set of differentiable vector fields on V. Such a manifold is called an almost paracontact Riemannian manifold, and its structure an almost paracontact Riemannian structure. A structure which satisfies only condition (1) is called a Sato structure. The following theorem is proved in [**4**].

THEOREM 1.1. *The almost product structure Φ induces on a hypersurface the Sato stucture ψ in the following way*

$$\Phi B = B\psi \oplus (\eta \otimes N), \quad \Phi N = B\xi,$$

where B is the differential of the immersion i Kinto $\mathcal{M}^n$.

Proof. $\Phi B = B\psi \oplus (\eta \otimes N), \qquad \Phi^2 BX = \Phi[B\psi \oplus (\eta \otimes N)]X,$

$$\begin{aligned} \Phi^2 BX &= \Phi[B\psi X \oplus \eta(X)N], \qquad BX = \Phi B(\psi X) + \eta(X)\Phi(N) \\ BX &= [B\psi \oplus \eta \otimes N](\psi X) + \eta(X)\Phi(N), \\ BX &= [B\psi^2(X) + \eta\psi(X)N] + \eta(X)\Phi(N) \\ BX &= B(X) - \eta(X)B\xi + 0 + \eta(X)B\xi, \qquad BX = BX \end{aligned}$$

and

$$\begin{aligned} \Phi N &= B\xi, \qquad \Phi^2 N = \Phi B\xi, \\ N &= (B\psi \oplus (\eta \otimes N))\xi, \qquad N = B\psi\xi + \eta(\xi)N, \qquad N = N. \end{aligned}$$

2. The structure $(\overline{\psi}, \xi, \eta, \overline{g})$, on K. We shall assume that rank $\varphi = n-1$. Then M is a 2-dimensional manifold. Let K be a hypersurface in $\mathcal{M}^n$ orthogonal on vector

$$N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \text{in } \mathcal{M}^n.$$

Let $\overline{\varphi}, \overline{m}$ and $\overline{g}$ be restrictions of the structure φ and tensors m and g on K, and let

$$\xi = \left.\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right\} n-1 \qquad \eta = \underbrace{(0, \dots, 0, 1)}_{n-1}.$$

$\overline{\varphi}, \overline{m}$ and $\overline{g}$ have matrixes of the form

$$\overline{\varphi} = \begin{bmatrix} E_h & 0 & 0 \\ 0 & -E_{2r-n-h} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \overline{m} = \begin{bmatrix} 0_h & 0 & 0 \\ 0 & 0_{2r-n-h} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \overline{g} = \begin{bmatrix} E_h & 0 & 0 \\ 0 & E_{2r-n-h} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

THEOREM 2.1. $\overline{\varphi}$ *is a Sato structure.*

Proof. Since $\overline{\varphi}^2 = 1 - \overline{m}$, multiplying the corresponding matrixes it is clear that $\overline{m} = \xi\eta$, $\overline{\varphi}^2 = I - \eta \otimes \xi$, $\overline{\varphi}\xi = 0$, $\overline{\varphi\eta} = 0$, $\xi(\eta) = 1$, and moreover:

THEOREM 2.2. $(\overline{\varphi}, \xi, \eta, \overline{g})$ *is an almost paracontact Riemannian structure on* K.

Proof. It is clear that $\eta(X) = \overline{g}(\xi, X)$, $\overline{g}(\overline{\varphi}X, \overline{\varphi}Y) = \overline{g}(X,Y) - \eta(X)\eta(Y)$ which prves the theorem.

In Theorem 1.1. it is proved that an almost product structure induces on a hypersurface a Sato structure. From this and from Theorems 2.1 and 2.2. we obtain the following:

THEOREM 2.3. *The almost product structure* $\Phi = 2\varphi^2 - 1$ *induces on* K *a structure Sato moreover an almost paracontact Riemannian structure.*

3. Relation between ψ and the $(\overline{\varphi}, \xi, \eta, \overline{g})$ structure. We shall examine what conditions must be satisfied so that the structure ψ induced by $\Phi = 2\varphi^2 - 1$ on K^* is equal to the structure $\overline{\varphi}$.

Let K^* be the subspace of K whose vectors have the form

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_h \\ 0_1 \\ \vdots \\ 0_{2r-n-h} \\ z_1 \end{bmatrix}$$

THEOREM 3.1. *The almost product structure* Φ *induces on* K^* *the Sato structure* $\overline{\varphi}$.

Proof. We shall prove the relations $\Phi B = B\overline{\varphi} \oplus (\eta \otimes N)$ and $\Phi N = B\xi$ on K^*. That $\Phi N = B\xi$ is clear using

$$\Phi = \begin{bmatrix} E_h & 0 & 0 & 0 \\ 0 & E_{2r-n-h} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

To prove the relation $\Phi B = B\overline{\varphi} \oplus (\eta \otimes N)$ on K^*, we shall prove $BX = \Phi B(\overline{\varphi}X) + \eta(X)\Phi(N)$ for the vectors $X \in K^*$.

Let $X \in K$, we obtain

$$BX = \begin{bmatrix} x_1 \\ \vdots \\ x_h \\ y_1 \\ \vdots \\ y_{2r-n-h} \\ z_1 \\ 0 \end{bmatrix}, \quad \Phi B(\overline{\varphi}X) + \eta(X)\Phi(N) = \Phi \begin{bmatrix} x_1 \\ \vdots \\ x_h \\ -y_1 \\ \vdots \\ -y_{2r-n-h} \\ 0 \\ 0 \end{bmatrix} + z_1 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_h \\ -y_1 \\ \vdots \\ -y_{2r-n-h} \\ z_1 \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_h \\ y_1 \\ \vdots \\ y_{2r-n-h} \\ z_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_h \\ -y_1 \\ \vdots \\ -y_{2r-n-h} \\ z_1 \\ 0 \end{bmatrix}$$

when $y_1 = 0, \ldots y_{2r-n-h} = 0$. From this it is easy to see that $\Phi B = B\overline{\varphi} \oplus (\eta \otimes N)$ only on the space K^*. This proves the Theorem.

Since $\overline{\varphi}$ and $\overline{g}$ satisfy the following on $K^* : \eta(X) = \overline{g}(\xi, X),\ g(\overline{\varphi}X, \overline{\varphi}Y) = g(X, Y) - \eta(X)\eta(Y)$, we have

THEOREM 3.2. *The almost product structure* Φ *induces on* K^* *the almost paracontant Riemannian structure* $(\overline{\varphi}, \xi, \eta, \overline{g})$.

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