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ON THE OPERATIONS OVER RELATIONS IN THE RELATIONAL MODEL OF DATA WITH TWO TYPES OF NULL VALUES

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Abstract. The relational model of data with two types of null values is considered. The basis for defining operations in such a model is three-valued logic; thus, three-valued relations of equality of tuples, relation membership, relation inclusion and equality, are introduced. Extended operations of relational algebra [**3**] are defined ("true" and "maybe" operations, applicable to date extended by null values). Properties of those operations, analogous to the properties of operations of the basic relational algebra, are proved. Since the ability to change the order in which specific operations are executed influences query optimization, extended operations are proved to have that property too.

1. Introduction

Extended relational model of data with null values was considered from different aspects in [**5,7**]. This paper attempts to define a generalization of Codd's model with only one type of null value [**5**], to the model with two types of null values. Null values considered are the same as in [**7**]: undefined value (ρ) and unknown value (ω). Undefined value has the meaning "property unapplicable". Unknown value stands for any value from the domain, but not for the undefined value, ie. ω is unknown but defined value.

With each operation of the relational algebra [**3**] we associate two operations: "true" and "maybe" operations. The result of "true" operation should contain tuples which are known to satisfy the condition defining the corresponding usual operation; result of "maybe" operation should contain tuples for which it is not known if they satisfy that condition.

We also define three valued relations of equality of tuples, relation membership and inclusion; the value set is $\{T, F, \omega\}$–"true", "false" and "unknown" ("maybe").

For our extended relations and operations we prove some properaties as well as some relationships between the operations, which are significant for query optimization.

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2. Relational model of data with two types of null values.

In the relational model the data are arranged into one or more multicolumn tables-called relations. Each column of the relations is labeled by distinct name of the "attribute" represented by that column, and each attribute takes values from the designated set called "domain" of that attribute. A row of a relation– a tuple– is related to an entity which is characterized by the values of that tuple.

Formally, given sets $D_1, D_2, \dots, D_n$, (domains, $n > 0$), n-ary relation R with the attributes $A_1, A_2, \dots, A_n$, defined on these domains (denoted $R(A_1, A_2, \dots, A_n)$), is a finite subset of the Cartesian product $\times\{D_i : i = 1, 2, \dots, n\}$.

As opposed to the mathematical notion of a relation, relations which are used here are time varying sets because some tuples may be deleted, inserted or modified. The row and column order in this dynamic relation is unimportant.

The need for dealing with the incomplete information in the datebase arised naturally, because some of the relevant data may be missing. The values, representing any kind of missing data are called "null" values. The two most important types of null values are "unknown" (value which belongs to some domain but at present is not known in the database), and "undefined" value–unapplicable property—for example the property "spouse's name" for someone who is not married. These two kinds of nulls are considered in [**7**] using the denotational semantics and approximation aspect, and the relations with "unknown" values are treated in [**5**] with the operations of the relational algebra. In this section we define the extended relational model with the unknown (ω) and the undefined (ρ) values. Mathematical notions of equality of types, relation membership and inclusion are defined as three valued relations. The set of truth values is $\{T, F, \omega\}$,—"true", "false" and "maybe"; "maybe" has the meaning of unknown truth value and stands for T of F. Logical operations $\neg, \wedge, \vee$ in this three valued logic are defined as in [**5**]:

$$\begin{array}{c|ccc} \neg & T & F & \omega \\ \hline & F & T & \omega \end{array} \qquad \begin{array}{c|ccc} \wedge & T & F & \omega \\ \hline T & T & F & \omega \\ F & F & F & F \\ \omega & \omega & F & \omega \end{array} \qquad \begin{array}{c|ccc} \vee & T & F & \omega \\ \hline T & T & F & T \\ F & T & F & \omega \\ \omega & T & \omega & \omega \end{array}$$

These three valued relations will provide a basis for defining operations of the extended relational algebra (in the next section).

We use the usual notation for relations, tuples and operations on relations and tuples ([**2–5**]); we also introduce the folowing notation and assumptions:

1. $\hat{x} = (x_1, x_2, \dots, x_n),\ x_1 = x_2 = \cdots = x_n = x,\ n \geq 0\ x \in D \cup \{\rho, \omega\}$ for any domain D;

2. $X = \{X_1, X_2, \dots, X_n\}$. where $X_1, X_2, \dots, X_n$ are all of the attributes of some relation R or tuple r;

3. $\tau(\psi)$ is the truth value of the statement $\psi; \tau(\psi) \in \{T, F, \omega\}$;

4. There is no attribute in any relation with all ρ values;

5. There is no relation with a tuple $\hat{\rho}$;

6. There are no two identical (character by character) tuples in a relation.

Extended equality of the individual elements of a tuple ($\doteq$) is defined by the following table:

$\tau(x \doteq y)$	$x \in D$	$x = \omega$	$x = \rho$
$y \in D$	$\tau(x = y)$	ω	F
$y = \omega$	ω	ω	F
$y = \rho$	F	F	T

D is any domain and $\tau(x = y)$ is the truth value of the ordinary equality ($\in \{T, F\}$).

Definition 1. (Equality of tuples). (i) Tuples $r(X), s(Y)$ are "true"-equal (T equal, $\tau(r \doteq s) = T$), iff they are "true" equal on each attribute from $X \cap Y$, and on the atributes from $X\Delta Y (X\Delta Y = (X\backslash Y) \cup (Y\backslash X))$, the tuple which takes value on them, has the value "ρ"; (ii) $r(X)$, $s(Y)$ are not equal $\tau(r \doteq s) = F$) iff there is an attribute from $X \cap Y$ on which they are not equal, or there is an attribute from $X\Delta Y$ on which any of the tuples has a value different from ρ; (iii) $r(X)$, $s(Y)$ are "maybe" equal (ω-equal, $\tau(r \doteq s) = \omega$), otherwise. Formally, $\tau(r \doteq s) = r[X\backslash Y] = \hat{\rho} \wedge s[Y\backslash X] = \hat{\rho} \bigwedge_{A \in X \cap Y} \tau(r[A] \doteq s[A])$; $r[Z]$ denotes a projection of a tuple r on attributes from Z.

The properties analogous to the reflexivity, symmetry and transitivity of the ordinary equality are applied to the defined relation.

Theorem 1. *For the tuples $r(X)$, $s(X)$, $t(Z)$, the following holds:*

1. $\tau(r \doteq r) \in \{T, \omega\}$; 2. $\tau(r \doteq s) = \tau(s \doteq r)$;

3.a. $\tau(r \doteq s) = T \wedge \tau(s \doteq t) = T \Rightarrow \tau(r \doteq t) = T$,

b. $\tau(r \doteq s) = \omega \wedge \tau(s \doteq t) = T \Rightarrow \tau(r \doteq t) = \omega$.

Proof. 1. and 2. follow from the definition of true and maybe equality. The fact that $\tau(r \doteq r)$ may be ω means that in two identical (character by character) tuples (r, r) which contain ω-values, some of the corresponding ω-values may be substituted by different values from the domain and also may be substituted by the same values. In the first case two tuples become different, and in the second– they become equal. Thus, it is unknown whether they are equal. Proof of 3 is in the Appendix.

Remark 1. More general transitivity does not hold, and we cannot extend the property 3. on both ω-equalities in the antecedent of the implication, i.e. the following does not hold: $\tau(r \doteq s) \in \{\omega, T\} \wedge \tau(s \doteq t) \in \{\omega, T\} \Rightarrow \tau(r \doteq t) \in \{\omega, T\}$; for example, $r = (2, \omega)$, $s = (\omega, 3)$, $t = (4, \omega)$, $\tau(r \doteq s) = \omega$ and $\tau(s \doteq t) = \omega$ but $\tau(r \doteq t) = F$.

For $\tau(r \doteq s) = T$ (or ω) we shall write also $r =_T s$ (or $r =_\omega s$).

In the definition of a relation membership we will use the partial order relationship "to be more informative" and "to be more equally informative"—for individual elements and tuples, denoted $\sqsupset$ and $\sqsupseteq$ respectively [**6**]. In every domain extended with $\omega, \rho, \rho \sqsupset a \sqsupset \omega$, for $a \in D$, D-any domain; $x \sqsupseteq y \overset{\text{def}}{\Leftrightarrow} x \sqsupset y \vee x = y$, for $x, y \in D \cup \{\omega, \rho\}$. For two n-tuples $r(X),\ s(X),\ r \sqsupseteq s \overset{\text{def}}{\Leftrightarrow} \bigwedge_{A \in X} (r[a] \sqsupseteq s[A])$.

The reason for defining the relationship $\sqsupseteq$ in such a way is the following: the value ω stands for any value from the domain, i.e. every value from the domain contains more information than the value ω. Using the same criterion, the value ρ contains more information than any value from the domain, because none of the values from the domain can take place of ρ without producing inconsistency (the value cannot be, simultaneously, defined and undefined).

The relation R equipped with $\sqsupseteq$ is not a lattice. Indeed, R contains neither the suppremum $(\hat{\rho})$ nor the infimum $(\hat{\omega})$ with respect to $\sqsupseteq$. Moreover if we adopt the assumption that every relation R has a primary key which does not take the null values [**4**], then every chain in R is represented by exactly one element, ie. there will not be n-tuples comparable with $\sqsupseteq$-relationship.

Definition 2. (Relation membership) Given an n-tuple r, let $G(r)$ be the set of all n-tuples which are more informative than r or equally informative as r, and which are "true" equal or "maybe" equal with r, i.e. $G(r) = \{s : s \sqsupseteq r \wedge \tau(s \doteq r) \in \{T, \omega\}\}$. Let $M(r)$ denote the set of the maximums of all the chains from $G(r)$, i.e. $M(r) = \{\max_{\sqsupseteq} L : L \text{ is a chain from } G(r)\}$. For a relation $R(X)$ and a tuple $r(Y)$,

(i) r is a "true" element of $R(\tau(r \mathrel{\dot\in} R) = T)$ iff every tuple $s \in M(r)$ is "true" equal with some tuple t from R;

(ii) r is "maybe" element of $R(\tau(r \mathrel{\dot\in} R) = \omega)$ iff it is "maybe" equal to a tuple t from R, and r is not "true"element of R;

(iii) r is not an element of $R(\tau(r \mathrel{\dot\in} R) = F$, otherwise. Formally,

$$\tau(r \in R) \overset{\text{def}}{=} (\bigwedge_{s \in M(r)} \bigvee_{t \in R} \tau(t \doteq s)) \bigvee_{t \in R} \tau(t \doteq r)$$

Remark 2. The definition of a "true" membership includes both the case when the tuple r does not have ω-values ($M(r) = r,\ \tau(r \doteq t) = T$ for some $t \in R$), and the case when the tuple r has ω-values, and, whatever combination of values from the corressponding domains substitutes all ω-values in r, the resulting tuple s (which is in $(M(r))$, is "true" equal to some of the tuples from R. For example,

$R(X_1 X_2)$,

0	a
1	a
1	b

$r(X_1 : \omega,\ X_2 : a,\ X_3 : \rho)$, domain $(X_1) = \{0, 1\}$; $\tau(r \mathrel{\dot\in} R) = T$; $M(r) = \{(0, a, \rho),\ (1, a, \rho)\}$.

Instead of $\tau(r \mathrel{\dot\in} R) = T(\tau(r \mathrel{\dot\in} R) = \omega)$, we shall also write $r \in_T R(r \in_\omega R)$, and for a negation $r \notin_T R(r \notin_\omega R)$.

Definition. 3. (Inclusion and equality of relations) Given relations $R(X)$, $S(Y)$, $\tau(R \dot{\subseteq} S) \stackrel{\text{def}}{=} \bigwedge_{r\in R} \tau(r \dot{\in} S)$; $\tau(\varnothing \dot{\subseteq} S) = T$ for every relations S; $\tau(R \doteq S) = \tau(R \dot{\subseteq} S) \wedge \tau(S \dot{\subseteq} R)$.

Insetad of $\tau(R \dot{\subseteq} S) = T$, we shall also write $R \subseteq_T S$; similarly for $\subseteq_\omega$.

Remark 3. For the relations without nulls (ω, ρ), the definitions 1–3 reduce to the definitions of the corresponding two-valued relations (equality, set membership, inclusion).

Remark. 4. For the equality of relations (Definition 3), the same theorem as for tuples –Theorem 1, holds.

3. Operations in the extended model

In this section we define to operations of the extended relational algebra. As in [**5**], for each operation from the basic relational model, we define two operations – "true" (T) and "maybe" (ω). The results of these operations consist of the tuples which, recpectively, exactly and maybe satisfy the conditions defining the operations.

Prior to defining the extended traditional set operations (union, intersection, difference), we define the restriction operations of one relation with respect to another.

Definition 4. (T, F, ω-restriction of a relation with respect to another relation): For the relations $R(X), S(Y)$,

$$R \mid_T S \stackrel{\text{def}}{=} \{r : r \in R \wedge r \in_T S\}; \ \ , \ R \mid_\omega S = \{r : r \in R \wedge r \in_\omega S\};$$
$$R \mid_F S = \{r : r \in R \wedge \tau(r \dot{\in} S) = F\}.$$

Remark 5. Restriction operation of one relation with respect to another is a kind of "semijoin" operation [**1**], where "semijoin" is performed on all the attributes of the relations. For example, $R \mid_T S \cup R \mid_\omega S$ could be represented using the semijoin, as $R\langle X \doteq Y \rangle S$, where " $\doteq$"means $=_T$or$=_\omega$. For a relation R without null values, $R \mid_T S$ is representable as $R\langle X =_T Y \sqsupset S$, and similarly for $R \mid_\omega S$. For a relation R with ω-values, a representation of $R \mid_T S$ using the semijoin is not immediate and has to be more complicated. The reason for this complication is in the fact that the definition of $R \mid_T S$ uses a relation membership and the definition of semijoin uses an equality of tuples; according to our definition 2, extended membership cannot be immediately substituted by the equality of some tuples, in a way it could be done for two-valued logic and relations without null values. For union compatible relations R, S [**3**] without null values, the operation $R \mid_T S$ reduces to the usual intersection $R \cap S$, and the operation $R \mid_F S$ reduces to the usual set difference $R \backslash S (R \mid_\omega S$ is empty).

For $R \mid_T R$ we will write R_T (and similarly for R_ω); these are, "true" and "maybe" subsets of R. It is easy to show, using Definition 4 and Theorem 1 that the following holds:

(*) $\quad (R_T)_T = R_T$ and $(R_\omega)_\omega = R_\omega$;

(**) $\quad r \in R \Rightarrow r \in R_T$ or $r \in R_\omega$;

(***) $\quad r \in_T R \Rightarrow r \in_T R_T$.

In the sequel, we make the assumption that there is no tuple which contains ω-values and which is "true" element of any of the relations; for example, it is not possible that we have:

$R(X_1\ X_2)$, domain $(X_1) = \{0, 1\}$.

ω	a
0	a
1	a

For the operation of restriction of one relation with respect to another it is easy to prove properties such as: the operation of T-restriction is distributive with respect to the union and intersection operations; the operation of F-restriction is distributive with respect to the union and intersection on the left, and with respect to them on the right it satisfies De Morgan's laws (in certain cases some of these assertions hold for "true" subsets of relations as operands).

Now, using the definitions 1—4, we define "true" and "maybe" (T and ω) traditional set operations (union, intersection, difference and Cartesian product):

Definition 5. (Difference, union, intersection, Cartesian product) For the relations $R(X), S(Y)$,

1. $R \backslash_T S \stackrel{\text{def}}{=} R_T \mid_F S$; $\quad R \backslash_\omega S \stackrel{\text{def}}{=} R \mid_\omega S \cup R_\omega \mid_F S$;

2. For $R \cup^* S = R \otimes \hat{\rho} \cup S \otimes \hat{\rho}$, where $\hat{\rho}$ is taken with the attributes leading to the union compatible relations [**3**], "true" and "maybe" union is:

$R \cup_T S \stackrel{\text{def}}{=} (R \cup^* S)_T$; $\qquad R \cup_\omega S \stackrel{\text{def}}{=} (R \cup^* S)_\omega$;

3. $R \cap_T S \stackrel{\text{def}}{=} (R_T \mid_T S)[X \cap Y]$; $\quad R \cap_\omega S \stackrel{\text{def}}{=} (R \mid_\omega S)[X \cap Y] \cup (S \mid_\omega R)[X \cap Y]$;

3. $R \times_T S \stackrel{\text{def}}{=} R_T \otimes S_T$; $\qquad R \times_\omega S \stackrel{\text{def}}{=} R \otimes S \backslash R \times_T S$.

Remark 6. In the definition of "true" and "maybe" intersection, the operation of projection on common attributes ($[X \cap Y]$) [**3**] is used.

For the defined operations it is possible to show (using the definitions, the properties of T, F restrictions of one relation with respect to another and the properties of "true" and "maybe" subsets of a relation) that the following theorem holds:

THEOREM 2.1. *The results of "true" and "maybe" operations are equal to their "true" and "maybe" subsets;*

2. $(R \cap_T S) \cup_T (R \backslash_T S) \subseteq R_T$*:*

3. *The operations* $\cup_T$, $\cap_T$, $\cup_\omega$, $\cap_\omega$ *are commutative,* $\cup_T$ *and* $\cap_T$ *are associative, and* $\cup_T$, $\cap_T$ *are distributive with respect to each other;*

4. *The operation* $\setminus_T$ *is distributive with respect to* $\cup_T$, $\cap_T$ *on the left, and with respect to them on the right, it satisfies De Morgan's laws;*

5. $(R \otimes S)_T = R \times_T S$ *and* $(R \otimes S)_\omega = R \times_\omega S$.

Remark 7. If $X \cap Y \neq \varnothing$ (in the definition of the Cartesian product), the common attributes have to be renamed in order to get the unique attribute names.

We now define some of the extended operations of the relational algebra, which are not traditional set operations but which are characteristic for manipulations with relations.

Definition 6. (θ-restriction) Given a relation $R(X)$, attributes A, $B \in X$, a constant $k \in$ domain (A) and $\theta \in \{=, <, \leq, > \geq, \neq\}$ $R[A\,\theta_T\,B] = \{r : r \in R \wedge r[A]\theta_T r[B]\}$; $R[A\,\theta_\omega B] = \{r : r \in R \wedge r[A]\theta_\omega r[B]\}$; a constant k may stands for B and $r[B]$ in this definition; $x\,\theta_T\,y$ is a notation for $\tau(x \cdot \dot{\theta} y) = T$ and similarly for $x\,\theta_\omega\,y$. For $\theta \in \{<, \leq, >, \geq\}$, $\dot{\theta}$-relationship is defined by the same table as $\doteq$-relationship (where θ stands for $=$), except for $\tau(\rho \mathrel{\dot{\lesseqgtr}} \rho) = F$; $\tau(x \neq y) = \neg(\tau(x \doteq y))$.

Remark 8. It seems more natural to define $\dot{\theta}$-relationship as a four-valued one with $\tau(\rho \mathrel{\dot{<}} a) = \rho$-undefined, instead of $\tau(\rho \mathrel{\dot{<}} a) = F$. However, since we define "true" and "maybe' operations, it is unimportant if some tuples do not enter in the result of these operations because it is "false" (F) that they satisfy some condition or because that condition is not even defined for them.

Definition 7. (θ-join) For the relations $R(X)$ and $S(Y)$, attributes $A \in X$ and $B \in Y$ and θ as in Definition 6,

$$R[A\theta_T B]S \stackrel{\text{def}}{=} (R \otimes S)[A\theta_T B]; \quad R[A\theta_\omega B]S \stackrel{\text{def}}{=} (R \otimes S)[A\theta_\omega B].$$

Remark 9. In the definitions 6, 7, for θ just "=", it may be, $A, B \subseteq X\,(Y)$, instead of A, $B \in X\,(Y)$.

Definition 7. Immediately implies the following property: $(R[A\theta_T B]S \cup R[A\theta_\omega B]S)[X] \subseteq R$ (analogous to θ-join property in ordinary model).

In the relational model without null values, a result of the natural join operation [**3**] is obtained from the result of $=$ join operation by projection; the same will hold for the operation of "true" natural join ($*_T$) in our model with two types of null values. In obtaining a result of "maybe" natural join ($\overset{*}{\omega}$) in the same way from a result of $=_\omega$ join operation, we wish to keep from every tuple $t \in R[A =_\omega B]S(t[A] =_\omega t[B])$, one of the projections on the corresponding pair of attributes from A, B, which is "more informative" than another or "equally informative" as another. Doing so, we save as much as possible of the "quantity of information" in the result of ω-join. For example, for the relations

$R\,(A$	B	$C)\,,$	$S\,(B$	C	$D)$
a	2	ω	ω	1	1
b	ω	1	ω	ω	1

$R[B, C =_\omega B, C]S(A\ B\ C\ B'\ C'\ D)$ and $R *_\omega S(A\ B\ C\ D)$.

A	B	C	B'	C'	D
a	2	ω	ω	1	1
a	2	ω	ω	ω	1
b	ω	1	ω	1	1
b	ω	1	ω	ω	1

A	B	C	D
a	2	1	1
a	2	ω	1
b	ω	1	1

For the first tuple $(a, 2, \omega, \omega, 1, 1)$ from $R[B, C =_\omega B, C]S$, among the projections on B, B' (and C, C'), we chose the projection on B (and C'), because pairs $(B : 2,\ C : \omega)$ and $(B' : \omega, C' : 1)$ can be equal only if both of them are equal to $(2, 1)$. If we choose a four-tuple $(a, 2, \omega, 1)$ to represent six-tuple $(a, 2, \omega, \omega, 1, 1)$ from $R[B, C =_\omega B, C]S$ in $R *_\omega S$-instead of $(a, 2, 1, 1)$, we loose the information that the former one may be in the result of $*_\omega$-join only if ω represents just 1.

To formalize what we said above, we introduce the following operation of an extended—selective—projection: let the attributes of a relation R be grouped in three groups $X = \{X_1, \ldots, X_i\}$, $Y = \{Y_1, \ldots Y_j\}$, $Z = \{Z_1, \ldots, Z_j\}$, some of which may be empty and some of which may contain the common attributes; the corresponding attributes from Y and Z have comparable domains. We define the projection of the relation R on the attribute set $\{X, J\}$, where $J_i = Y_i$ or $J_i = Z_i$ (for $i = 1, 2, \ldots, j$), depending upon which one of the projections $r[Y_i]$, $r[Z_i]$, for every tuple r from R, is "more or equally informative" (and rename J-attributes Y, again) in the following way:

$$R[X, [Y \vee Z)] = \{\sup\{r[X, Y], r[(X, Z]\} : r \in R \wedge r[Y] =_\omega r[Z]\};$$

supremum is taken with respect to the relationship $\sqsupseteq$.

Now we deduce, using projection and selective projection, the result of the extended natural join operation from the result of the extended =-join operation in the following way:

Definition 8. (Natural join) For the relations $R(X), S(Y)$ and the sets of attributes $A \subseteq X$, $B \subseteq Y$,

$$R[A *_T B]S = (R[A =_T B]S)[X, Y \backslash B];$$
$$R[A *_\omega B]S = (R[A =_\omega B]S)[(X \backslash A) \cup (Y \backslash B),\ (A \vee B)].$$

Remark 10. When $A = B = X \cap Y$, the "true" and "maybe" operations of natural join are written as $R *_T S$ and $R *_\omega S$.

The following theorem is analogous to the corresponding inclusion in the model without null values:

THEOREM 3. 1. $\tau(((R[A *_T B]S)[X])_T \mathrel{\dot{\subseteq}} R_T) = T$;

2. $\tau((R[A *_T B]S \cup R[A *_\omega B]S)[X] \mathrel{\dot{\subseteq}} R) \in \{T, \omega\}$.

Proof of theorem is in the Appendix.

Definition 9. (Division) For the relations $R(X)$, $S(Y)$, let $g_R(t)$ be, as in [**3**], the set $\{s : (t^\frown s) \in R\}$; for $A \subseteq X,\ B \subseteq Y$, "true" and "maybe" quotients are:

$$R[A \div_T B]S \overset{\text{def}}{=} \{t : t \in R[X \backslash A] \wedge S[B] \subseteq_T g_R(t)\};$$
$$R[A \div_\omega B]S \overset{\text{def}}{=} \{t : t \in R[X \backslash A] \wedge S[B] \subseteq_\omega g_R(t)\}.$$

The operation defined in named "division" because it has the following property (analogous to that in the basic model):

THEOREM 4. 1. $(R \times_T S) \div_T S_T = R_T$ (for $S_T \neq \varnothing$);
2. $(R \times_\omega S) \div_\omega S_\omega = R$ (for $S_\omega \neq \varnothing$);
3. $(R \times_\omega S) \div_T S_T = R_\omega$ (for $S_T \neq \varnothing$).

Proof of the theorem uses the following lemma:

LEMMA 1. $(R_1 \cup R_2)[A \div_T B]S = R_1[A \div_T B]S \cup R_2[A \div_T B]S$, if $S \neq \varnothing$, *and* R_1, R_2 *are union compatible relations [***3***],* $R_1[X \backslash A] \cup R_2[X \backslash A] = \varnothing$*; analogously for the operation* $\div_\omega$.

Proofs of Lemma 1 and Theorem 4 are in the Appendix.

Remark 11. All other combinations of "true" and "maybe" operations of Cartesian product and division, and "true" and "maybe" subsets of S (for example $(R \times_T S) \div_T S)$ do not give interesting results; they are either $\varnothing$ or they depend on the relationship between S, S_T, S_ω.

Remark 12. All the defined operations of the extended relational algebra, when applied to the relations without null values, reduce to the operations for the basic relational model [**3**].

Now, using the operations of the relational algebra and the extended relational algebra, we can express every query in the relational database with two types of null values. For example, let us have the databese with the following relations: LECTURES (Students # Subject Profesor) PROFESSORS (Name Position)

Students #	Subject	Profesor
n_1	A	X_1
n_2	A	ω
n_3	A	X_3
n_1	B	X_2
n_2	B	ω

Name	Position
X_1	assoc. prof.
X_2	professor
X_3	assit. prof.
X_4	asist. prof.
X_5	profesor

The query: find the number of all students who are positively (or maybe) taking lectures on the subject "A" from assistant professors, we express as:

(PROFESSORS[Position$=_T$ "assist. pr."] $*_T$ LECTURES [Subject$=_T$ "A"]) [Student #]

(PROFESSORS[Position$=_{T,\omega}$ "assist. pr."] $*_{T\omega}$ LECTURES [Subject$=_{T,\omega}$ "A"]) [Stud.#]

where, in the second expression, not all of the operations (restrictions and join) are T-operations

4. On the order in which the operations are performed

For the operations of the relational algebra, it may be very important in which order the operations are performed over the operand relations, if the result is to be obtained in the most effective way. For example, if we have to perform the operations of the Cartesian product over two relations and the restriction over one of them, it is significant that the restriction may precede the Cartesian product, because the product is a very expensive operation and it is desirable to perform it over as small sets as possible.

For the extended operations on the relational algebra, we exhibit some possibilities of changing order in performing the operations.

THEOREM 5. *For the relations* $R(X)$, $S(Y)$ *and the attributes* $A, B \in X$ *or* $A, B \subseteq X$, $C \subseteq X$, $D \subseteq Y$,

1a. $(R \times_T S)[A\,\theta_T\,B] = (R[A\,\theta_T\,B]) \times_T S, \quad X \cap Y = \varnothing;$

1b. $(R \otimes S)[A\,\theta_T\,B] = (R[A\,\theta_T\,B]) \otimes S;$

1c. $(R \times_\omega S)[A\,\theta_\omega B] = (R[A\,\theta_\omega\,B]) \times_\omega S;$

2a. $(R[A\,\theta_T\,B])[C] = (R[C])[A\,\theta_T\,B];$

2b. *analogous for* θ_ω*-restriction* $(A,\, B \in X)$

3a. $(R[C *_T D]S)[A\,\theta_T\,B] = (R[A\,\theta_T\,B])[C *_T D]S;$

3b. $(R[C *_\omega D]S)[A\,\theta_\omega\,B]) \subseteq (R[A\,\theta_\omega\,B])[C *_\omega D]S.$

Proof of this theorem is in the Appendix.

Any constant $c \in$ domain (A) in Theorem 5 may stand for B. The following example shows that the converse inclusion in 3b. does not hold:

$R(\underset{a}{A}\,\underset{1}{B}\,\underset{\omega}{C}), S(\underset{\omega}{B}\,\underset{2}{C}\,\underset{b}{D}), (R[\{B,C\} *_\omega \{B.C\}]S)[B =_\omega C] = T(\underset{a}{A}\,\underset{1}{B}\,\underset{2}{C}\,\underset{b}{D})[B =_\omega C] = \varnothing$
$(\equiv T1)$ and
$(R[B =_\omega C])[\{B,C\} *_\omega \{B,C\}]S = R_1(\underset{a}{A}\,\underset{1}{B}\,\underset{\omega}{C})[\{B,C\} *_\omega \{B,C\}]S \equiv T2(\underset{a}{A}\,\underset{1}{B}\,\underset{2}{C}\,\underset{b}{D});$
$T1 \subset T2$ but $T2 \not\subset T1$.

5. Conclusion

In this paper we presented the relational model of data with two types of null values. Null values considered in this model are undefined value (ρ) and unknown value (ω). We assumed that the ω-value is unknown but defined value. The given definitions can be easily modified in order to extend ω to include the undefined value ρ, too, so that the properties proved in the paper still hold.

The results presented in the paper are:

(i) Extended three-valued relations of the equality of tuples, relation membership and inclusion are defined. We proved the properties for the extended equality of tuples ($\doteq$), which are analogous to the reflexivity, symmetry and transitivity of the usual two-valued equality (Theorem 1). The definition of these extended relations was found to be the common need when defining the extended operations of the relational algebra.

(ii) With some operations of the relational algebra we associated two operations—"true" and "maybe" (T and ω) operations. Their results contain, respectively, tuples which are known to satisfy the condition of the corresponding usual operation, and for which it is unknown whether they satisfy that condition. For these operations we exhibited the properties analogous to the properties of the corresponding usual operations of the relational algebra (theorems 2, 3, 4).

(iii) For the extended operations of the Cartesian product, restriction, natural join and projection, it is shown that the order in performing these operations can be changed (Theorem 5). This property could be very useful in query optimization.

This is only one aspect of the relational data model with two types of null values. In the paper following this one we present another aspect of the extended relational model with two types of null values—functional and multivalued dependencies and their extended properties.

Appendix

Proof of Theorem 1.3. In this proof, a. will indicate the case a, and b.–the case b from the formulation of the theorem.

$$\text{Since } \tau(r \doteq s) = \begin{cases} \text{a. } T \\ \text{b. } \omega \end{cases} \wedge \tau(s \doteq t) = T, \text{ the following holds } ((1)-(4)):$$

(1) $r[X\backslash Y] = \hat{\rho} \wedge s[Y\backslash X] = \hat{\rho}$,

(2) $\begin{cases} \text{a. } (\forall A \in X \cap Y)(r[A] = s[A] \wedge s[A] \neq w) \\ \text{b. } (\forall A \in X \cap Y)(\tau(r[A] \doteq s[A] \neq F) \wedge (\exists A \in X \cap Y)(\tau(r[A] \doteq s[A]) \neq T), \end{cases}$

(3) $s[Y\backslash Z]\hat{\rho} \wedge t[Z\backslash Y] = \hat{\rho}$ and

(4) $(\forall A \in Y \cap Z)(s[A] = t[A])$; (1)—(4) imply (1') and (2'): item$(1')$ $t[X\backslash(Y \cap Z)] = \hat{\rho}$ and $t[Z\backslash(X \cap Y)] = \hat{\rho}$ (it is because (1) and (3) imply $s[Y\backslash(X \cap Z)] = \hat{\rho}$, which with (2) and (4) imply $r[(X \cap Y)\backslash Z] = \hat{\rho}$ and $t[(Y \cap Z)\backslash X] = \hat{\rho}$, and the corresponding set equalities between the attribute sets on which we project r, s, t, hold)

$(2')$ $\begin{cases} \text{a. } (\forall A \in X \cap Y \cap Z)(r[A] = t[A] \wedge r[A] \neq \omega \\ \text{b. } (\forall \in X \cap Y \cap Z)(\tau(r[A] \doteq t[A]) \neq F \wedge (\exists A \in X \cap Y \cap Z)(\tau(r[A] \doteq t[A]) \neq T) \end{cases}$

From $(1')$ we obtain, by projection, $(1'')$ and $(1''')$:

$(1'')$ $r[X\backslash Z] = \hat{\rho}$ and $t[Z\backslash X] = \hat{\rho}$;

$(1''')$ $r[X \cap Z)\backslash Y] = t[(X \cap Z)\backslash Y] = \hat{\rho}$; $(1''')$ and $(2')$ imply $(2'')$:

$$(2'')\begin{cases} a\cdot (\forall A\in X\cap Z)(r[A]=t[A]\wedge r[A]\neq\omega) \\ b\cdot (\forall A\in X\cap Z)(\tau(r[A]\doteq t[A]\neq F)\wedge(\exists A\in X\cap Z)(\tau(r[A]\doteq t[A])\neq T); \end{cases}$$

From (1") and (2") we obtain $\tau(r\doteq t)=\{a.\ Tb.\omega$.

Proof of Theorem 3.

1. $r\in((R[A*_T B]S)[X])+T$ implies that $r\in((R[A*_T B]S)[X])$ and $r\in_T(R[A*_T B]S)[X]$; now from definitions 7,8, $r\in R$ and $r\in_T R$, i.e. $r\in R_T$; it implies (using $(*)$), $\tau(r\mathrel{\dot\in} R_T)=T$;

2. $r\in(R[A*_T B]S\cup R[A*_\omega B]S)[X]$ implies (using Definition 8) $r\in R)$ or $\exists s\in R)(\exists t\in S)(r[X\backslash A]=s[X\backslash A]\wedge r[A]=\sup\{s[A],t[B]\}\wedge s[A]=_\omega t[B])$; according to Definiton 1, $r\in R$. or $(\exists s\in R)(r=_\omega s)$, and by Definition 2, we have $\tau(\dot\in R)\in\{T,\omega\}$.

Proof of Lemma 1. $t\in(R_1\cup R_2)[A\div_T B]S$ implies that $t\in(R_1\cup R_2)[X\backslash A]\wedge S[B]\subseteq_T g_{R_1\cup R_2}(t)$;

$\Rightarrow(t\in R_1[X\backslash A]\vee t\in R_2[X\backslash A])\wedge S[B]\subseteq_T\{u:(t\frown u)\in R_1\cup R_2\}$

$\Rightarrow(t\in R_1[X\backslash A]\vee t\in R_2[X\backslash A])\wedge S[B]\subseteq_T\{u:(t\frown u)\in R_1\}\vee S[B]\subseteq_T$ $\{u:(t\frown u)\in R_2\})$ (because $R_1[X\backslash A]\cap R_2[X\backslash A]=\varnothing$)

$\Rightarrow(t\in R_1[X\backslash A]\wedge S[B]\subseteq_T g_{R_1}(t))(t\in R_2[X\backslash A]\wedge S[B]\subseteq_T g_{R_1}(t))$-all other cases are impossible, because of the hypothesis of lemma;

$\Rightarrow t\in R_1[A\div_T B]S\vee t\in R_2[A\div_T B]S$, ie. $t\in R_1[A\div_T B]S\cup R_2[A\div_T B]S$;

Convese inclusion is trivial; quite analogously for the operation $\div_\omega$.

Proof of Theorem 4.

1. $(R\times_T S)\div_T S_T=\{t:t=r\frown s\wedge r\in R_T\wedge s\in S_T\}\div_T S_T$
$=\{r:r\in R_T\wedge S_T\subseteq_T g_{R\times_T S}(r)\}=\{r:r\in R_T\wedge S_T\subseteq_T S_T\}=R_T$;

2. $(R\times_\omega S)\div_\omega S_\omega=\{t:t=r\frown s\wedge r\in R\wedge s\in S\wedge(r\in R_\omega\vee s\in S_\omega)\}\div_\omega S_\omega$
$=\{t:t=r\frown s\wedge((r\in R_T\wedge s\in S_\omega)\vee(r\in R_\omega\wedge s\in S))\}\div_\omega S_\omega$
$=$ (Lemma 1)$\{t:t=r\frown s\wedge r\in R_T\wedge s\in S_\omega\}\div_\omega S_\omega$
$\cup\{t:t=r\frown s\wedge r\in R_\omega\wedge s\in S\}\div_\omega S_\omega$
$=\{r:r\in R_T\wedge S_\omega\subseteq_\omega S_\omega\}\cup\{r:r\in R_\omega\wedge S_\omega\subseteq_\omega S_\omega\}=R_T\cup R_\omega=R$;

3. $(R\times_\omega S)\div_T S_T=$ (Lemma 1)$\{t:t=r\frown s\wedge r\in R_T\wedge s\in S_\omega\}\div_T S_T$
$\cup\{t:t=r\frown s\wedge r\in R_\omega\wedge s\in S\}\div_T S_T$
$\{r:r\in R_T\wedge S_T\subseteq_T S_\omega\}\cup\{r:r\in R_\omega\wedge S_T\subseteq_T S\}=\varnothing\cup R_\omega=R_\omega$.

Proof of Theorem 5.

1a. $(R\times_T S)[A\theta_T B]=(R_T\otimes S_T)[A\theta_T B]=\{t:t\in R_T\otimes S_T\wedge t[A]\theta_T t[B]\}$
$=\{t:t=r\frown s\wedge r\in R_T\wedge s\in S_T\wedge r[A]\theta_T r[B]\}=\{r:r\in R_T\wedge r[A]\theta_T r[B]\}\otimes S_T$
$=\{r:r\in R_T\wedge r[A]\theta_T r[B]\}_T\otimes S_T$ (none of the tuples from R_T takes ω-values)

$= (R[A\theta_T B])_T \otimes S_T = (R[A\theta_T B]) \times_T S$;

1b. analogously to 1a;

1c. $(R \times_\omega S)[A\theta_\omega B] = \{t : t \in R \otimes S \wedge t \notin R_T \otimes S_T \wedge t[A]\theta_\omega t[B]\} = \{t : t \in R \otimes S \wedge t[A]\theta_\omega t[B]\}$($t \notin R_T \otimes S_T$ because $t[A]\theta_\omega t[B]$, which means that t takes ω-values)

$= \{t : t = r\hat{\ }s \wedge r \in R \wedge s \in S \wedge r[A]\theta_\omega r[B]\} = \{r : r \in R \wedge r[A]\theta_\omega r[B]\} \otimes S$

$= (R[A\theta_\omega B] \otimes S = ((R[A\theta_\omega B]) \otimes S)\backslash((R[A\theta_\omega B])_T \otimes S_T)$ (the set $(R[A\theta_\omega B])_T$ is empty because all tuples from $R[A\theta_\omega B]$ contain ω-values)

$= (R[A\theta_\omega B] \times_\omega S$;

2a. $(R[A\theta_T B])[C] = \{r : r \in R \wedge r[A]\theta_T r[B]\}[C] =$

$$\{u : u \in R[C] \wedge u[A]\theta_T u[B]\} = (R[C])[A\theta_T B];$$

2b. analogously to 2a;

3a. $(R[C *_T D]S)[A\theta_T B] = ((R[C =_T D]S)[X, Y\backslash D])[A\theta_T B]$

$= (((R \otimes S])[C =_T D])[X, Y\backslash D])[A\theta_T B]$

$= (((R \otimes S])[C =_T D])[A\theta_T B])[X, Y\backslash D]$ (because of 2a.)

$= (((R \otimes S])[A\theta_T B])[C =_T D])[X, Y\backslash D]$ (commutativity of conjuction)

$= (((R[A\theta_T B]) \otimes S][C =_T D])[X, Y\backslash D]$ (because of 1b.)

$= ((R[A\theta_T B])[C =_T D]S)[X, Y\backslash D] = (R[A\theta_T B])[C *_T D]S$ (definitions $7, 8)\}$;

3b. $t \in (R[C *_\omega D]S)[A\theta_\omega B]$ means, by definition 8, that

$t \in ((R[C =_\omega D]S)[(X\backslash C) \cup (Y\backslash D),\ (C \vee D)][A\theta_\omega B]$; it is equivalent to

(1) $(\exists r \in R)(\exists s \in S)(r[C] =_\omega s[D] \wedge t[X\backslash C] = r[X\backslash C]] \wedge t[C] =$

$= \sup\{r[C]s[D]\}) \wedge (t[A]\theta_\omega t[B]$; (1) implies that

$(t[A] = r[A] \vee t[A] \vee \sqsupseteq r[A]) \wedge (t[B] = r[B] \vee t[B] \sqsupseteq r[B])$, especially

(2) $t[A] \sqsupseteq r[A] \wedge t[B] \sqsupseteq r[B]$; since $t[A]\theta_\omega t[B]$ implies

(3) $(t[A] = \omega \wedge t[B] \neq \rho) \vee (t[A] \neq \rho \wedge t[B] = \omega)$, from (2) and (3) we have

$(r[A] = \omega \wedge r[B] \neq \rho) \vee (r[A] \neq \rho \wedge r[B] = \omega)$ and it exactly means (by definition of θ_ω) that

(4) $r[A]\theta_\omega r[B]$. (1) and (4) imply $t \in (R[A\theta_\omega B])[C *_\omega D]S$.

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