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ON THE DEGRE OF L_1-APROXIMATION BY MODIFIED BERNSTEIN POLYNOMIALS

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Abstract. Recently many researchers like Bojanić & Shisha, and A. Grundmann have obtained the degree of L_1 approximation to integrable functions by modified Bernstein polynomials. The object of the present note is to improve their results.

1. Introduction. The modified Bernstein polynomials

$$(P_n f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_{I_k} f(t)\,dt \tag{1.1}$$

Where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $I_k = [k/(n+1), (k+1)/(n+1)]$ were introduced by Kantorovič to approximate Lebesque integrable functions in L_1 norm. Using a weight function $W(x) = \sqrt{x(1-x)}$, Bojanič and Shisha have recently obtained the following rate of convergence in terms of integral modulus of continuity,

$$w(f,\delta)_{L1} = \sup_{|h|\le\delta} \int_0^1 \mid f(x+h) - f(x) \mid \, dx$$

by operators (1.1).

THEOREM 1. *Let f be a Lebesque integrable function in* $[0,1]$. *Then for* $n \ge 2$,

$$\int_0^1 W(x) \mid (P_{n-1} f)(x) - f(x) \mid \, dx \le 2\pi/3 \cdot w(f; 1/\sqrt{n})_{L1}. \tag{1.2}$$

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Later on, A. Grundmann [**2**] has given the following result.

THEOREM 2. *Let* f *be a Lebesgue integrable function in* $[0,1]$. *Then for* $n \geq 2$,

$$\int_0^1 |\,(P_{n-1})f(x) - f(x)\,|\; dx \leq 4w(f;1/\sqrt{n})_{L_1}. \tag{1.3}$$

The object of the present note [**4**] is to improve the results (1.2) and (1.3) because not much is known about the degree of L_1 approximation to integrable functions by modified Bernstein polynomials.

2. We need the following lemmas to prove our theorem.

LEMMA 2.1. *Let* $f_\delta(y) = 1/\delta \cdot \int_0^\delta f(y+t)\,dt \quad (\delta > 0)$, *then*

$$\|f_\delta^{(1)}(y)\|_{L_1} \leq 1/\delta \cdot w(f;\delta)_{L_1} \ \ \textit{and}\ \ \|f - f_\delta\|_{L_1} \leq w(f;\delta)_{L_1}.$$

Proof. The proof follows from [**2**].

LEMMA 2.2. *For* $0 \leq y \leq 1$ *and* $0 \leq r \leq n-1$, *one has*

$$(1 - r/n)p_{n,r}(y) \leq \sqrt{3}/8\sqrt{n}.$$

Proof. We know that

$$\begin{aligned}(n-r)/\sqrt{n}\cdot p_{n,r}(y) &\leq \binom{n-1}{r}(r+1)^{r+1}(n-r)^{n-r}\sqrt{n}(n+1)^{-n-1}, \ (0 \leq r \leq n-1)\\ &\leq \binom{n-1}{[n/2]}([n/2]+1)^{[n/2]+1}(n-[n/2])^{n-[n/2]}\sqrt{n}(n+1)^{-(n+1)} = U_n \quad \text{(Say)}\end{aligned}$$

where $[n/2]$ denotes the largest integer not exceeding $n/2$.

Case 1. When $n = 2m+1$, $(m \geq 1)$. Then

$$\begin{aligned}U_n &= \binom{2m}{m}(m+1)^{2m+2}\sqrt{2m+1}/(2m+2)^{2m+2} = \binom{2m}{m}\sqrt{2m+1}/2^{2m+2}\\ &= \frac{1\cdot 3\cdot 5\cdot\dots\cdot(2m-1)}{2\cdot 4\cdot 6\cdot\dots\cdot 2m}\cdot\frac{\sqrt{2m+1}}{4}\\ &= 1/4\sqrt{(1-1/4)(1-1/4\cdot 2^2)(1-1/4\cdot m^2)} \leq 1/4\cdot\sqrt{3/4} = \sqrt{3}/8 \ \ (m \geq 1).\end{aligned}$$

Case 2. When $n = 2m$, $(m \geq 1)$. Then

$$\begin{aligned}U_n &= (2m-1)!\sqrt{2m}m^m(m+1)^{m+1}/(m-1)!m!(1+2m)^{2m+1}\\ &= \frac{1\cdot 3\cdot\cdots\cdot(2m-1)}{2\cdot 4\cdot\cdots\cdot 2m}\cdot\frac{\sqrt{2m+1}}{4}\cdot\frac{m^{m+1/2}(m+1)^{m+1}}{(1/2+m)^{2m+3/2}}.\end{aligned}$$

Since $m^{2m+1}(1+m)^{2m+2} \leq (m+1/2)^{4m+3}$ for $m \geq 1$, we have $U_n = U_{2m+1} \leq \sqrt{3/8}$ by case 1.

LEMMA 2.3. *Let* $I_1 = \int_0^1 \mid (P_{n-1}f_y)(x) - f_y(x) \mid\, dx$. *Then for* $n \geq 2,\ 0 \leq y \leq 1,\ I_1 \leq 23/20\sqrt{n}$, *where*

$$f_y(x) = \begin{cases} 1 & \textit{when } x \leq y \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We get clearly $I_1 = I_2 + I_3$ (say), where

$$\begin{aligned}
I_2 &= \int_0^y \mid P_{n-1}f_y(x)(x) - f_y(x) \mid\, dx\\
&= \int_0^y \left| \sum_{k=0}^{n-1} p_{n-1,k}(x) - n\sum_{k=0}^{n-1} p_{n-1,k}(x) \int_{k/n}^{k+1/n} f_y(du)\right| dx\\
&= \int_0^y \left| \sum_{k=0}^{n-1} p_{n-1,k}(x) - \sum_{k=0}^{[ny]} p_{n-1,k}(x)\right| dx = \sum_{k=[ny]+1}^{n-1} \int_0^y p_{n-1,k}(x)\, dx,\\
I_3 &= \int_y^1 \mid P_{n-1}f_y(x)(x) - f_y(x) \mid\, dx = \sum_{k=0}^{[ny]}\int_y^1 p_{n-1,k}(x)\, dx\\
&= \sum_{k=0}^{[ny]}\left\{\int_0^1 p_{n-1,k}(x)dx - \int_0^y p_{n-1,k}(x)\, dx\right\}\\
&= \int_0^y\left\{1 - \sum_{k=0}^{[ny]} p_{n-1,k}(x)\right\} dx + ([ny]) + 1 - ny)/n\\
&\leq \sum_{k=[ny]+1}^{n-1}\int_0^y p_{n-1,k}(x)\, dx + 1/n
\end{aligned}$$

Using $\int_0^y p_{n-1,k}(x)dx = 1/n \cdot \sum_{l=k+1}^{n} p_{n,l}(y)$ we get easily that

$$
\begin{aligned}
I_1 &\le 2/n \cdot \sum_{k=[ny]+1}^{n-1} \sum_{l=k-1}^{n} p_{n,l}(y) + 1/n \\
&= 2/n \cdot \sum_{k=[ny]+1}^{n} \{k - [ny] - 1\} p_{n,k}(y) + 1/n \le 2 \sum_{k=[ny]+1}^{n} (k/n - y) p_{n,k}(y) + 1/n \\
&= 2\binom{n-1}{[ny]} y^{[ny]+1}(1-y)^{n-[ny]} + 1/n \le \sqrt{3}/4\sqrt{n} + 1/n \ \text{(Using Lemma 2.2)} \\
&\le 1/\sqrt{n} \cdot (\sqrt{3}/4 + \sqrt{2}/2) \le 23/20\sqrt{n}, \ (n \ge 2)
\end{aligned}
$$

This completes the proof

We prove the following result.

Theorem 3. *If is a Lebesque integrable function in* $[0,1]$ *then for* $x \ge 2$,

$$\int_0^1 \mid (P_{n-1}f)(x) - f(x) \mid \, dx \le 63/20 \cdot w(f; 1/\sqrt{n})_{L_1}. \tag{2.4}$$

Proof. Using $\|P_{n-1}\|_{L_1} = 1$, it can be obtained that

$$
\begin{aligned}
\|(P_{n-1}(f) - f)\|_{L_1} &\le \|P_{n-1}(f - f_\delta)\|_{L_1} + \|P_{n-1}(f_\delta) - (f_\delta))\|_{L_1} + \|(f_\delta - f)\|_{L_1} \\
&\le 2\|(f_\delta - f)\|_{L_1} + \|(P_{n-1}(f_\delta) - (f_\delta))\|_{L_1}.
\end{aligned} \tag{2.5}
$$

We can show similarly to what is done in [**3**] that

$$\|(P_{n-1}(f_\delta) - (f_\delta))\|_{L_1} \le \int_0^1 \mid f_\delta^{(1)}(y) R_n \mid (y)\, dy$$

$$\text{where} \qquad R_n(y) = \int_0^1 \mid (P_{n-1} f_x(y)(x) - f_x(y) \mid \, dx.$$

Clearly $R_n(y) = \int_0^1 \mid (P_{n-1} f_y(x)(x) - f_y(x) \mid \, dx$, [because $f_x(y) = 1 - f_y(x)$ everywhere, except for the point $x = y$ and $(P_{n-1}1)(x) = 1$]. So

$$R_n(y) = I_1 \le 23/20\sqrt{n}, \ n \ge 2, \ 0 \le y \le 1, \ \text{(Using Lemma 2.3)}. \tag{2.6}$$

Again using Lemma 2.1 and the result (2.6) in (2.5), we get

$$
\begin{aligned}
\|(P_{n-1}(f) - f\|_{L_1} &\le 2w(f;\delta)_{L_1} + 23/20\sqrt{n} \cdot \int_0^1 \mid f_\delta^{(1)}(y) \mid \, dy \\
&\le 2w(f,\delta)_{L_1} + 23/20\sqrt{n} \cdot \int_0^1 \mid f_\delta^{(1)} y \mid \, dy \le (2 + 23/20\delta\sqrt{n}) w(f,\delta)_{L_1}.
\end{aligned}
$$

Finaly choosing $\delta = 1/\sqrt{n}$, we get the required result (2.4). This completes the proof.

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