PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 34 (48), 1983, pp. 239–242

ON TWO OPEN PROBLEMS OF CONTRACTIVE MAPPINGS

V. Totik

Abstract. Two open problems are solved concerning the fixed points of contractive mappings. The first is an example of a shrinking mapping of the closed unit ball in a Banach space without any fixed point. The second solves a question of B. Fischer.

1. Let (X, d) be a metric space, $T : X \to X$ a mapping of X into itself. T is said to be shrinking if $d(Tx, Ty) < d(x, y)$ for every $x, y \in X$.

It is well known (see e.g. [**3**]) that if X is compact and $T : X \to X$ is a shrinking mapping, then T has a fixed point. By a beautiful theorem of Browder [**1**] the same conclusion holds provided X is the closed unit ball of a Hilbert space and T is shrinking. In connection with these results D. R. Smart raised the following question [**3**, p. 39]: "Does every shrinking mapping of the closed unit ball in a Banach space have a fixed point?" The aim of this paragraph is to give negative answer to this problem.

THEOREM 1. *There exists a Banach space B and an affine shrinking mapping T of the closed unit ball U of B into the boundary ∂U of U such that T does not have any fixed point.*

Proof. Let $c_0 = \{x = \{x_i\}_{i=1}^{\infty} \mid \lim_{i\to\infty} x_i = 0\}$ be the space real sequences converging to 0 with norm $\|x\| = \sup_i \mid x_i \mid$. Let $B = c_0$ and $T(x_1, x_2, \dots, x_n, \dots) = (1, x_2/2 + 1/2,\ 2x_2/3 + 1/3, \dots,\ (1-1/n)x_n + 1/n, \dots)$ i.e. T is defined by $(Tx)_n) = (1-1/n)x_n + 1/n$. If U is the unit ball in B, then clearly $T : U \to \partial U$ and T is affine. T is shringing. Let $x = \{x_i\}_1^{\infty}$, $y = \{y_i\}_1^{\infty}$, $x \neq y$. Then

$$0 < \varepsilon := \|x - y\| = |x_{n_0} - y_{n_0}|$$

for some n_0. Let $N > 2$ be so large that the inequalities

$$|x_n| < \varepsilon/4, \ |y_n| < \varepsilon/4 \qquad (n \geq N)$$

AMS Subject Classification (1980): Primary 54 H 25, 47 H 10.

be satisfied. Now

$$|(Tx)_i - (Ty)_i| = (1 - 1/i)|x_i - y_i| \le \begin{cases} 2\,\varepsilon/4 = \varepsilon/2 & \text{if } i \ge N \\ (1 - 1/N)|x_i - y_i| \le |1 - 1/N|\varepsilon & \text{if } i < N \end{cases}$$

i.e.

$$\|Tx - Ty\| \le (1 - 1/N)\varepsilon,$$

and so T is really a shrinking mapping.

Finally T does not have any fixed point: if $x = \{x_i\}_1^\infty$ where a fixed point of T, then we would have

$$(1 - 1/i)x_i + 1/i = (Tx)_i = x_i$$

i.e. $x_i = 1$ for all i, but the sequence $\{1\}_1^\infty$ does not belong to $B = c_0$.

We have proved our theorem.

2. In [**2**] B. Fischer made the following conjecture. Suppose S and T are mapping of the complete matrix space X into itself, with either S or T continuous, satisfying the inequality

$$d(Sx, TSy) \le c\,\mathrm{diam}\{x, Sx, Sy, TSy\} \tag{1}$$

for all x, y in X, where $0 \le c < 1$. Then S and T have a unique common fixed point.

This conjecture has been open even for compact X. Now we show that it is true for $c < 1/2$ but false for $c \ge 1/2$.

THEOREM 2. *If X is complete, $S : X \to X$, $T : X \to X$ with property* (1), *where $c < 1/2$, then S and T have a unique common fixed point. On the other hand, there are a four point X and $S : X \to X$, $T : X \to X$ mappings of X without fixed point satisfying*

$$d(Sx,\ TSy) \ge 1/2\ \ \mathrm{diam}\ \{x,\ Sx,\ Sy,\ TSy\}.$$

Thus, if $\alpha < 1/2$ we do not need any continuity assumption, and for $\alpha \ge 1/2$ even the simultaneous continuity of S and T and the compactness of X do not help.

Proof. To prove the first part of our theorem let $x_0 \in X$ be arbitrary and let

$$x_n = \begin{cases} (TS)^{n/2}x_0, & \text{if } n \text{ is even} \\ S(TS)^{(n-1)/2}x_0, & \text{if } n \text{ is odd.} \end{cases}$$

By (1)

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(STSx_{2n-2}, TSx_{2n-2}) \le c\,\mathrm{diam}\{Sx_{2n-2},\ TSx_{2n-2}, STSx_{2n-2}\} = \\ &= c\,\mathrm{diam}\{x_{2n-1},\ x_{2n},\ x_{2n+1}\} \le c(d(x_{2n},\ x_{2n-1}) + d(x_{2n+1},\ x_{2n})) \end{aligned}$$

and thus

$$(2) \qquad d(x_{2n+1}, x_{2n}) \le (c/(1-c))d(x_{2n}, x_{2n-1}) \qquad (n \ge 1)$$

Similarly,

$$d(x_{2n+2}, x_{2n+1}) = d(Sx_{2n}, TSx_{2n}) \le c \ \operatorname{diam}\{x_{2n}, x_{2n+1}, x_{2n+2}\} \le \\ \le c(d(x_{2n+1}, x_{2n}) + d(x_{2n+2}, x_{2n+1}))$$

by which

$$(3) \qquad d(x_{2n+2}, x_{2n+1}) \le (c/(1-c))d(x_{2n+1}, x_{2n})$$

Since $c < 1/2$ we have $c/(1-c) < 1$, and so (2) and (3) imply that the sequence x_n is a Cauchy sequnce and thus, by completeness, $x_n \to z(n \to \infty z \in X)$. Using again (1) we get

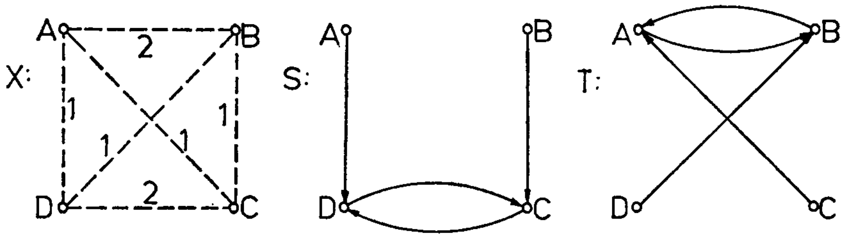
$$d(Sz, x_{2n+2}) \le c \ \operatorname{diam}\{z, Sz, x_{2n+1}, x_{2n+2}\} \le \\ c(d(Sz, z) + d(z, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}))$$

Letting here $n \to \infty$ we obtain $d(Sz, z) \le cd(Sz, z)$ i.e. $d(Sz, z) = 0$, $Sz = z$. But then

$$d(z, Tz) = d(Sz, TSz) \le c \ \operatorname{diam}\{z, Sz, TSz\} = c\, d(z, Tz)$$

i.e. $d(z, Tz) = 0$, $Tz = z$ and thus z is a common fixed point of S and T. The uniqueness of the common fixed point follows easily from (1).

After this let us prove that the conjecture is false for $c = 1/2$ and hence also $c \ge 1.2$. Let $X = \{A, B, C, D\}$ with $d(A, D) = d(B, C) = d(B, D) = 1$ and $d(A, B) = d(C, D) = 2$ (see the first figure) and let S and T be the two mapping indicated below:



Neither S nor T have any fixed point. However, $Sx \in \{D, C\}$, $TSy \in \{A, B\}$ and so $d(Sx, TSy) = 1$ for every $x, y \in X$; furthermore

a) $d(x, Sx) = 2$, if $x = C$ or $x = D$

b) $d(Sx, Sy) = 2$, if $x + A$ and $y \in \{B, D\}$ or $x = B$ and $y \in \{A, C\}$

c) $d(x, TSy) = 2$, if $x = A$ and $y \in \{A, C\}$or $x = B$ and $y \in \{B, D\}$, i.e. in any case $\operatorname{diam}\{x, Sx, Sy, TSy\} = 2$ and so (1) holds for every $x, y \in X$ with $c = 1/2$.

We have proved our theorem.

REFERENCES

[1] F. E. Browder, *Fixed point theorems for noncompact mappings in a Hilbert space*, Proc. Nat. Acad. Sci. U.S.A., 53, 1272-6.

[2] B. Fischer, *Results and a conjecture on fixed points*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **62** (1977) 769–775.

[3] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, 1974.

Szeged, Bolyai Institute
Aradi vétanuk tere 1
Hungary

(Received 11 07 1981)