THE LEVITZKI RADICAL FOR $\Omega$-GROUPS

A. Buys and G. K. Gerber

Abstract. The concept locally nilpotent ideal of an $\Omega$-group is defined. The class of locally nilpotent $\Omega$-groups is a Kurosh-Amitsur radical class. Furthermore, the Levitzki radical of an $\Omega$-group is the intersection of all $\Omega$-prime ideals $P$ such that $G/P$ is Levitzki semi-simple.

1. Notations and definitions. The notation and definitions of Higgins [4] and Buys and Gerber [2] will be used. For the sake of convenience we define the basic concepts. By $a = (a_1, a_2, \ldots, a_n) \in G$ we mean that $a_i \in G$, $i = 1, 2, \ldots, n$. Higgins [4] called words which involve only the operations $\omega \in \Omega$, monomials. We shall call such words $\Omega$-words. If $f(x) = f(x_1, x_2, \ldots, x_n)$ is an $\Omega$-word in the indeterminates $x_1, x_2, \ldots, x_n$ then $f(a) = f(a, a, \ldots, a)$. Let $\Omega$ be a fixed set of operations.

1.1 Definition. $\omega \in \Omega$ will be called a trivial operation in the variety $K$ of $\Omega$-groups if $x\omega = 0$ is satisfied in $K$. That is for all $G \in K$ and $a \in G$, $a\omega = 0$ holds. $\omega \in \Omega$ is a non-trivial operation if it is not trivial. An $\Omega$-word which involves only non-trivial operations will be called a non-trivial $\Omega$-word.

In Buys and Gerber [2], we defined the concept of an $\Omega$-prime ideal for an $\Omega$-group. That definition should actually be:

1.2 Definition. An ideal $P$ of the $\Omega$-group $G$ is called an $\Omega$-prime ideal if for all non-trivial $\omega \in \Omega$ and ideals $A_1, A_2, \ldots, A_n$ of $G$ such that $A_1 \ldots A_n\omega^G \subseteq P$ it follows that $A_i \subseteq P$ for some $i = 1, 2, \ldots, n$. All the results of Buys and Gerber [2] carries over with this slight alteration.

2. Locally nilpotent $\Omega$-groups. Bhandari and Sexana [1] called an ideal $I$ of a near-ring $N$ locally nilpotent if any finite subset of $I$ is nilpotent. They have shown that their definition coincides with the well-known definition of Levitzki defined for associative rings.

*AMS Subject Classification (1980): Primary 20N99, Secondary 16A12, 16A22, 08A99.*
2.1 Definition. A subset $S$ of the $\Omega$-group $G$ is nilpotent if there exists a non-trivial $\Omega$-word $f(x)$ such that $f(S) = \{f(s) | s \in S\}$ is zero.

2.2 Definition. Let $A$ be a subset of the $\Omega$-group $G$. $A$ is called locally nilpotent if any finite subset of $A$ is nilpotent.

2.3 Corollary. If $A \subseteq B \subseteq G$ and $B$ is locally nilpotent then $A$ is locally nilpotent. If $A \subseteq G$ is nilpotent then $A$ is locally nilpotent.

2.4 Lemma. Let $I$ be an ideal of the $\Omega$-group $G$. $G$ is locally nilpotent if and only if $I$ and $G/I$ are locally nilpotent.

Proof. From 2.3 it follows that $I$ is locally nilpotent. Let $\{g_1 + I, g_2 + I, \ldots, g_n + I\}$ be any finite subset of $G/I$. Since $G$ is locally nilpotent there exists a non-trivial $\Omega$-word $f(x)$ such that $f(a) = 0$ for all $a \in \{g_1, g_2, \ldots, g_n\}$. It follows that

$$f(a + I) = f(a) + I \quad \text{(Higgins [4, Theorem 3A])}$$

$$= I \quad \text{for all } a + I \in \{g_1 + I, g_2 + I, \ldots, g_n + I\}.$$ Thus $G/I$ is locally nilpotent.

For the converse let $\{g_1, g_2, \ldots, g_n\}$ be any finite subset of $G$. Since $G/I$ is locally nilpotent, there exists a non-trivial $\Omega$-word $f(x)$ such that $f(a) = 0$ for all $a + I \in \{g_1 + I, g_2 + I, \ldots, g_n + I\}$. It follows that $f(a) \in I$ for all $a \in \{g_1, g_2, \ldots, g_n\}$. Let $A = \{f(a) | a \in \{g_1, g_2, \ldots, g_n\}\}$. $A$ is finite. Since $A \subseteq I$ there exists a non-trivial $\Omega$-word $f_1(y)$ such that $f_1(b) = 0$ for all $b \in A$. In particular, $f_1(f(a)) = 0$ for all $a \in \{g_1, g_2, \ldots, g_n\}$. Since $f_1(f(x))$ is a non-trivial $\Omega$-word, the lemma follows.

2.5 Lemma. Let $I$ and $J$ be locally nilpotent ideals of the $\Omega$-group $G$. $I + J$ is a locally nilpotent ideal of $G$.

Proof. The lemma follows from Higgins [4, Theorem 3C] and 2.4.

2.6 Corollary. A finite sum of locally nilpotent ideals of $G$ is a locally nilpotent ideal of $G$.

2.7 Lemma. If $I_\alpha, \alpha \in A$, are locally nilpotent ideals of $G$ then $\sum I_\alpha$ is a locally nilpotent ideal of $G$.

Proof. Since any finite subset of $\sum I_\alpha$ is contained in a finite sum of locally nilpotent ideals, the result follows from 2.6.

2.8 Theorem. The class $\mathcal{G} = \{G | G$ is a locally nilpotent $\Omega$-group$\}$ is an absolutely hereditary radical class.

Proof. Properties R3, R5 and R7 of Rjabuhin [5] respectively follow from 2.4, 2.7 and 2.4. From Rjabuhin [5], Theorem 1.2 it follows that $G$ is a radical class.
From 2.3 it follows that \( G \) is an absolutely hereditary class (Rjabuhi [5, Definition p. 151]).

2.9 THEOREM. Let \( L(G) \) be the Levitzki radical of \( G \) that is \( L(G) \) is the sum of all locally nilpotent ideals of \( G \). \( L(G) = \cap \{ P_\alpha | P_\alpha \) is an \( \Omega \)-prime ideal of \( G \) such that \( L(G/P_\alpha) = 0 \).

Proof. Every locally nilpotent ideal in \( G \) and thus also \( L(G) \) is contained in \( P_\alpha \) for each \( \Omega \)-prime ideal \( P_\alpha \) with \( L(G/P_\alpha) = 0 \). It follows that \( L(G) \subseteq \cap \{ P_\alpha | P_\alpha \) is an \( \Omega \)-prime ideal of \( G \) such that \( L(G/P_\alpha) = 0 \) = \( P \) (say).

Assume there exists an \( a \in P \) such that \( a \notin L(G) \). Since \( a \notin L(G) \) every ideal \( I \) of \( G \) such that \( a \in I \) is not locally nilpotent. This holds for \( a \in G \). Thus there exists an \( A = \{ a_1, a_2, \ldots, a_n \} \subseteq a \) such that \( A \) is not nilpotent. Furthermore, \( \{ f(a) | a \in A \} \) is not nilpotent for any nontrivial \( \Omega \)-word \( f(x) \). Otherwise there would exist a non-trivial \( \Omega \)-word \( f_1(y) \) such that \( f_1(\{ f(a) | a \in A \}) = 0 \) and thus \( f_1(f(a)) = 0 \) for all \( a \in A \) contradicting the fact that \( A \) is not nilpotent. Let \( J = \{ I \} \) be an ideal of \( G \) such that \( L(G) \subseteq I \) and \( \{ f(a) | a \in A \} \subseteq I \) for any non-trivial \( \Omega \)-word \( f(x) \). \( J \neq \emptyset \) since \( L(G) \in J \). Applying Zorn’s lemma \( J \) has a maximal element \( Q \) (say). Thus \( L(G) \subseteq Q \) and \( \{ f(a) | a \in A \} \subseteq Q \) for any non-trivial \( \Omega \)-word \( f(x) \). We show that \( Q \) is an \( \Omega \)-prime ideal with \( L(G/Q) = 0 \). We need only show that \( G/Q \) is an \( \Omega \)-prime \( \Omega \)-group (Buys and Gerber). Let \( \omega \in \Omega \) be non-trivial and \( I_1/Q, I_2/Q, \ldots, I_n/Q \) ideals of \( G/Q \) such that \( (I_1/Q I_2/Q \ldots I_n/Q) \omega = 0 \). From Higgins it follows that \( I_1 I_2 \ldots I_n \omega \subseteq Q \). If \( I_j/Q \neq 0 \) for each \( j = 1, 2, \ldots, n \) then \( I_j \supseteq Q \). Since \( Q \) is maximal there exist non-trivial \( \Omega \)-words \( f_1(x_1), f_2(x_2), \ldots, f_n(x_n) \) such that \( \{ f(a) | a \in A \} \subseteq I_j \) \( j = 1, 2, \ldots, n \). Therefore

\[
\{ f_1(a) | a \in A \} \ldots \{ f_n(a) | a \in A \} \omega \subseteq I_1 I_2 \ldots I_n \omega \subseteq Q.
\]

In particular we have \( (f_1(a) f_2(a) \ldots f_n(a)) \omega \in Q \) for each \( a \in A \). Thus there exists a non-trivial \( \Omega \)-word \( g(x) \) defined by \( g(x) = (f_1(x) f_2(x) \ldots f_n(x)) \) such that \( \{ g(a) | a \in A \} \subseteq Q \). This is a contradiction. It follows that \( I_j/Q = 0 \) for some \( j \) and thus that \( G/Q \) is an \( \Omega \)-prime \( \Omega \)-group.

Suppose that \( W/Q \neq 0 \) is a locally nilpotent ideal of \( G/Q \). Then \( W \supseteq Q \) and there exists a non-trivial \( \Omega \)-word \( f(x) \) such that \( \{ f(a) | a \in A \} \subseteq W \) since \( Q \) is maximal. The family of cosets \( \{ f(a) + Q | a \in A \} \) is a finite set in \( W/Q \). Since \( W/Q \) is locally nilpotent, \( \{ f(a) + Q | a \in A \} \) is nilpotent. Thus there exists a non-trivial \( \Omega \)-word \( f_1(x) \) such that \( f_1(b) = 0 \) for every \( b \in \{ f(a) + Q | a \in A \} \). It follows that \( \{ f_1(f(a)) | b \subseteq A \} \subseteq Q \) which is a contradiction. Therefore \( L(G/Q) = 0 \). We have proved that \( Q \) is one of the ideals \( P_\alpha \) such that \( L(G/P_\alpha) = 0 \) and, therefore \( P \subseteq Q \). But \( A \subseteq P \) and \( \{ f(a) | a \in A \} \subseteq P \) for every \( \Omega \)-word and in particular for non-trivial \( \Omega \)-words. Since \( P \subseteq Q \) it also holds for \( Q \) and this is a contradiction. Therefore \( P \subseteq L(G) \).

As a result of the definition of Rjabuhi [5, p. 156], we have

2.10 THEOREM. Every Levitzki semi-simple \( \Omega \)-group is isomorphic to a subdirect sum of \( \Omega \) prime Levitzki semi-simple \( \Omega \)-groups.
REFERENCES


Department of Mathematics
University of Port Elizabeth
6000 Port Elizabeth
South Africa

(Received 12 11 1982)