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ASYMPTOTIC BEHAVIOUR OF FOURIER TRANSFORMS IN $\mathbf{R}^n$

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Abstract. If a function defined on a cone in the n-dimensional Euclidean space $\mathbf{R}^n$ is regularly varying at zero, then its Fourier transform is regularly varying at infinity.

1. Introduction. The following one-dimensional theorem is well known (see [**1**] and [**2**]). Let the function f be integrable and even or odd on the real line and let its Fourier transform, defined by

$$\widehat{f}(t) = \int_0^\infty f(x) \cos tx dx \ \text{ or } \ \widehat{f}(t) = \int_0^\infty f(x) \sin tx dx$$

be monotone decreasing. Then for $0 < \alpha < 1$ we have

$$f(x) \sim x^{-\alpha} L(1/x), \ \ x \to 0 \Longleftrightarrow \widehat{f}(t) \sim C_\alpha t^{\alpha-1} L(t), \ \ t \to \infty$$

where L is a slowly varying function and C_α is a constant.

We shall prove that an analogous statement holds in n dimensions.

The set $\Gamma \subset \mathbf{R}^n$ is a *cone* if $x \in \Gamma$ implies $\lambda x \in \Gamma$, for every $\lambda > 0$. We shall always assume that Γ is closed, convex, and that it has a nonempty interior. The *dual cone* of the cone Γ is defined by $\Gamma^* = \{x \in \mathbf{R}^n : x \cdot y \geq 0, \forall y \in \Gamma\}$. We shall also assume that $\operatorname{int} \Gamma^*$ is nonempty. For these definitions see [**5**], [**6**] or [**4**].

In the present paper we deal with the Fourier transforms of regularly varying functions.

The Fourier transform of a function F which is Lebesgue integrable on Γ is defined by

$$(1) \qquad \widehat{F}(x) = \int_\Gamma F(t) e^{ix \cdot t} dt.$$

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The real part and the imaginary part of this integral are called the *cosine* and *sine transform* respectively

$$\widehat{F}_c(x) = \int\limits_{\Gamma} F(t)\cos x \cdot t dt \quad \widehat{F}_s(x) = \int\limits_{\Gamma} F(t)\sin x \cdot t dt.$$

We shall consider cosine transforms only, the results about sine transforms being similar, and we shall also write $\widehat{F}$ instead od $\widehat{F}_c$.

Regularly varying functions in several variables were defined by Yakymiv [**7**], who generalized the well known definition of Karamata. Here we shall give a variant of this definition and we refer the reader to [**7**] and [**4**] for more detail and for some properties of regularly varying functions.

Let $r : \mathbf{R}_+ \to \mathbf{R}_+$ denote a regularly varying function in one variable (i.e. $\lim\limits_{\lambda\to\infty} r(\lambda\mu)/r(\lambda) = \mu^\rho$, for some $\rho \in \mathbf{R}$, called the index of r, and for every $\mu > 0$). Let Γ be a cone in $\mathbf{R}^n$. A measurable function $R : \Gamma \to \mathbf{R}_+$ is said to be regularly varying (at infinity) if there is a regularly varying function in one variable r and a function $\varphi : \Gamma \to \mathbf{R}_+$ such that

$$\lim_{\lambda\to\infty} \frac{R(\lambda x)}{r(\lambda)} = \varphi(x) \tag{2}$$

uniformly in $x \in B$, for all compact sets $B \subset \Gamma \setminus \{0\}$.

We shall always write R for a regularly varying function and r and φ for the functions that correspond to R as in (2). The function φ is continuous and homogeneous of order ρ, i.e. $\varphi(\lambda x) = \lambda^\rho \varphi(x)$, for all $\lambda > 0$ and $x \in \Gamma$, where ρ is the index of r.

A measurable function $R : \Gamma \to \mathbf{R}_+$ *is regularly varying at zero* (with index ρ) if the limit

$$\lim_{\lambda\to\infty} \frac{R(x/\lambda)}{r(\lambda)} = \varphi(x)$$

exists uniformly in $x \in B$, for every compact set $B \subset \Gamma \setminus \{0\}$.

The characteristic function of Γ will be denoted by θ_Γ (or simply θ). If F is a function defined on Γ, its convolution with θ is called the primitive of F (with respect to Γ) and is denoted by $\mathcal{I}_\Gamma F(x) = F * \theta(x) = \int\limits_{\Gamma\cap x-\Gamma} F(t)dt$.

If Γ and G are two cones, we shall say that G is *smaller* than Γ, or that Γ is larger than G, if $G \subset \operatorname{int}\Gamma$ (recall that all cones are assumed to be closed).

Now we can state the main theorem of the paper.

Theorem 1. *Let $0 < \rho < n$. Let R be integrable on a cone Γ, and regularly varying at zero with index ρ, i.e. let*

$$\lim_{\lambda\to\infty} \frac{R(u/\lambda)}{r(\lambda)} = \varphi(u), \quad u \in \Gamma \tag{3}$$

uniformly in compact sets from $\Gamma\setminus\{0\}$. *If the cosine Fourier transform* R *is positive in a cone* G^* *which is smaller than* Γ^*, *then*

$$\lim_{\lambda\to\infty}\frac{\mathcal{I}_{G^*}\widehat{R}(\lambda x)}{r(\lambda)}=\mathcal{I}_{G^*}\widehat{\varphi}(\lambda x),\quad x\in G^*, \tag{4}$$

uniformly in compact sets in $G^*\setminus\{0\}$, *i.e. the primitive of the Fourier transform is regularly varying at infinity in* G^* *with index* ρ.

Observe that for homogeneous functions φ the Fourier transform $\widehat{\varphi}$ cannot be defined by formula (1), since no homogeneous function is integrable on Γ. We deal with this problem in Section 2. In Section 3 we shall prove Theorem 1. In Section 4 we obtain the asymptotic behavior of the function $\widehat{R}$, under the additional assumption that it is monotone. In Section.5 we consider the dual case when the function R itself (and not) its Fourier transform is monotone. Theorem 2 shows that in this case too we have an analogue of Theorem 1. The combination of these two theorems gives in the one dimensional case the theorem of Aljančić, Bojanić and Tomić we mentioned at the beginning. However, in the n-dimensional case we cannot expect a statement of the "if and only if" type, since for a function defined on a cone we obtain the asymptotic behavior of its Fourier transform on the dual cone only, and have no information about its behavior on the rest of $\mathbf{R}^n$.

In the one-dimensional case it happens that the Fourier transform is supported by the dual cone; for example, the Fourier transform of an even function is even. A certain analogy with this case could be obtained in the n-dimensional space if we consider cones Γ that are contained an even number of times in $\mathbf{R}^n$ (i.e., the union of translations of Γ is equal to $\mathbf{R}^n$). An example of such a cone is $\mathbf{R}_+$. We can extend a function supported in Γ to the whole space $\mathbf{R}^n$ by continuing it in an "even" or "odd" way. The dual cone Γ^* is also contained an even number of times in $\mathbf{R}^n$, and the Fourier transform of f is also "even" or "odd" in $\mathbf{R}^n$. In this case, the values of $\widehat{f}$ in $\mathbf{R}^n$ are determined by the values of $\widehat{f}$ in Γ^*, and $\widehat{f}$ and f are interchangeable.

2. Fourier transforms of homogeneous functions. In this section we shall compute the Fourier transform of homogeneous functions. We shall assume that the homogeneous function φ defined on a cone Γ is continuous, and that $0<C_1<\varphi(x')<C_2$ for some constants C_1 and C_2 and all $x'=x/|x|$ from the intersection of Γ with the unit sphere $\Sigma\subset\mathbf{R}^n$. In other words, φ is like the function appearing in (2).

Let Γ be a cone in $\mathbf{R}^n$. The *tube domain* T^Γ with base Γ is the set of all $z=x+iy\in\mathbf{C}^n$ (the n-dimensional complex Euclidean space) such that $y\in\operatorname{int}\Gamma$. The *Laplace transform* of a measurable function F on Γ is defined by

$$\mathcal{L}F(z)=\int_\Gamma F(t)e^{iz\cdot t}dt=\int_\Gamma F(t)e^{-y\cdot t}e^{ix\cdot t}dt,\quad z\in T^{\Gamma^*}, \tag{5}$$

(it is the Fourier transform of the function $F(y)e^{-y\cdot t}$); see [**5**] or [**6**].

It was proved in [**6**] that when the function F is a temperate distribution (= locally integrable in Γ and of polynomial growth in infinity) the Laplace transform $\mathcal{L}F(z)$ is analytic in T^{Γ^*}. For $z = x \in \mathbf{R}^n$ $(y = 0)$ the Laplace transform (5) reduces to the Fourier transform. The problem is that the tube domain T^{Γ^*} in which $\mathcal{L}F$ is defined does not contain $\mathbf{R}^n$. However, if φ is homogeneous, $\mathcal{L}\varphi$ has an analytic continuation in a domain containing the cone $\operatorname{int}\Gamma^* \subset \mathbf{R}^n$. This will be proved in Proposition 1 and after that the Fourier transform $\widehat{\varphi}(x)$ will be defined as $\mathcal{L}\varphi(x)$, for $x \in \operatorname{int}\Gamma^*$.

PROPOSITION 1. *Let $\rho > -n$. Let $\varphi : \Gamma \to \mathbf{R}_+$ be continuous and homogeneous of order ρ. Then*

a) *the following representation holds*

$$\mathcal{L}\varphi(z) = I^{\rho+n}\Gamma(\rho+n)^{1)} \int\limits_{\Sigma\cap\Gamma} \frac{\varphi(t')}{(z\cdot t')^{\rho+n}}dt', \quad z \in T^{\Gamma^*} \tag{6}$$

b) *$\mathcal{L}\varphi(z)$ has an analytic continuation in the domain*

$$D = \mathbf{C}^n \setminus \bigcup_{\sigma\in\Sigma\cap\Gamma} \{z : z\cdot\sigma = 0\}.$$

Proof. Since $\rho > -n$, the function φ is locally integrable in Γ and thus, by a remark above, $\mathcal{L}\varphi(z)$ is analytic in T^{Γ^*}. On the other hand, $z \in T^{\Gamma^*}$, $t' \in \Sigma\cap\Gamma$ implies $y\cdot t' > 0$; consequently, $z\cdot t' = x\cdot t' + iy\cdot t'$ is never zero, and thus the integral on the right-hand side of (6) is also an analytic function in T^{Γ^*}. Therefore, to prove (6) it is sufficient to prove that this equality holds for $z = iy$, $y \in \operatorname{int}\Gamma^*$ (i.e. for $x = 0$). But for $x = 0$, by introducing the polar coordinates $r = |t|$, $t' = t/|t|$ in the integral defining $\mathcal{L}\varphi$, we have

$$\begin{aligned}\mathcal{L}\varphi(iy) &= \int\limits_{\Gamma} \varphi(t)e^{-y\cdot t}dt = \int\limits_{\Sigma\cap\Gamma} dt' \int\limits_0^\infty \varphi(rt')e^{-r(y\cdot t')}r^{n-1}dr \\ &= \int\limits_{\Sigma\cap\Gamma} \frac{\varphi(t')dt'}{(y\cdot t')^{\rho+n}} \int\limits_0^\infty r^{\rho+n-1}e^{-r}dr = i^{\rho+n}\Gamma(\rho+n) \int\limits_{\Sigma\cap\Gamma} \frac{\varphi(t')dt'}{(iy\cdot t')^{\rho+n}}\end{aligned}$$

which proves part a).

b) It is easily seen that the integral $\int\limits_{\Sigma\cap\Gamma} \varphi(t')/(z\cdot t')dt'$ is indeed analytic in D (which is larger than T^{Γ^*}). This follows from the fact that the denominator $z\cdot t'$ is never zero in D. This completes the proof of the proposition.

[1)] Here Γ denotes, of course, the gamma function (defined by $\Gamma(\alpha) = \int\limits_0^\infty e^{-t}t^{\alpha-1}dt$) which has nothing in common with the cone Γ, except for the unfortunate collision of notation.

Now we define the Fourier transform of φ by

$$\widehat{\varphi}(x) = \mathcal{L}\varphi(x) = i^{\rho+n} \int\limits_{\Sigma\cap\Gamma} \frac{\varphi(t')dt'}{(iy\cdot t')^{\rho+n}}, \quad x \in \operatorname{int}\Gamma^* \tag{7}$$

and Proposition 1 shows that $\widehat{\varphi}$ is well defined, since $\operatorname{int}\Gamma^* \in D$.

Observe that $\widehat{\varphi}$ is homogeneous of order $-\rho - n$ in $\operatorname{int}\Gamma^*$.

As a special case consider $\widehat{\varphi} = \theta_{\Gamma^*}$ (on the cone Γ^*). The Laplace transform of θ_{Γ^*} is called the Cauchy kernel of the cone Γ

$$\mathcal{K}_\Gamma(z) = \mathcal{L}\theta_{\Gamma^*}(z) = \int\limits_{\Gamma} e^{iz\cdot t}dt, \quad z \in T^\Gamma. \tag{8}$$

Formula (6) applied to this function yields

$$\mathcal{K}_\Gamma(z) = i^n\Gamma(n) \int\limits_{\Sigma\cap\Gamma^*} \frac{dt'}{(z\cdot t')^n}, \quad z \in T^\Gamma. \tag{9}$$

This was proved in [**6**], and Proposition 1 is an fact a simple generalization of this special case.

On the other hand, if we take $\varphi(t) = |t|^\rho$ and $\Gamma = \mathbf{R}_+$, formula (6) coincides with a well known result:

$$\widehat{\varphi}(x) = |x|^{-\rho-n}2^{\rho+n}\Gamma((\rho+n)/2)\Gamma(-\rho/2)$$

(see [**5**]. This is easily seen if we use the fact that

$$\int\limits_{\Sigma} dt'|x'\cdot t'|^{\rho+n} = 2\pi^{(n-1)/2}\Gamma((-\rho-n+1)/2)\Gamma(-\rho/2) \quad \text{(see [3])}.$$

Let us note some properties of the Cauchy kernel.

LEMMA 1. *Let $\mathcal{K}_\Gamma(x+iy)$, $x \in \mathbf{R}^n$, $y \in \Gamma$ be the Cauchy kernel of the cone Γ. Then*

(i) $\mathcal{K}_\Gamma(\lambda z) = \lambda^{-n}\mathcal{K}_\Gamma(z), \quad z = x+iy$

(ii) $|\mathcal{K}_\Gamma(x+iy)| \le \mathcal{K}_\Gamma(iy), \quad y \in \operatorname{int}\Gamma$

(iii) $|\mathcal{K}_\Gamma(x+iy)| \le \mathcal{K}_\Gamma(x), \quad x \in \operatorname{int}\Gamma$

(iv) $|\mathcal{K}_\Gamma(x)| = i^n\mathcal{K}_\Gamma(ix), \quad x \in \operatorname{int}\Gamma$

(v) $|\mathcal{K}_\Gamma(x')| < C, \quad x' \in \Sigma\cap\Gamma_1,$

where Γ_1 is smaller than Γ.

Proof. (i) Follows easily, by a change of variables in the integral defining the Cauchy kernel (8)

$$\mathcal{K}_\Gamma(\lambda z) = \int\limits_{\Gamma^*} e^{i\lambda z\cdot t}dt = \lambda^{-n}\int\limits_{\Gamma^*} e^{iz\cdot u}du = \mathcal{K}_\Gamma(z).$$

(ii) Also strightforward:

$$|\mathcal{K}_\Gamma(x+iy)| = \left|\int_{\Gamma^*} e^{-y\cdot t} e^{ix\cdot t} dt\right| \le \left|\int_{\Gamma^*} e^{-y\cdot t} dt\right| = \mathcal{K}_\Gamma(iy)$$

(iii) By formula (9) we have,

$$|\mathcal{K}_\Gamma(x+iy)| = \Gamma(n)\left|\int_{\Sigma\cap\Gamma^*} \frac{dt'}{(x\cdot t' + iy\cdot t')^n}\right| \le \Gamma(n)\int_{\Sigma\cap\Gamma^*} \frac{dt'}{|x\cdot t'|^n}.$$

Now, if $x \in \Gamma$, then $x \cdot t' > 0$, so that from the preceding formula it follows that

$$|\mathcal{K}_\Gamma(x+iy)| \le \Gamma(n)\int_{\Sigma\cap\Gamma^*} \frac{dt'}{(x\cdot t')^n} = |\mathcal{K}_\Gamma(x)|, \quad x \in \Gamma,$$

where for the last equality we have again used (9), which is valid for $x \in \Gamma$ by Proposition 1 b).

(iv) Follows by an application of (9) for $x \in \Gamma$.

(v) Since $x' \in \Sigma\cap\Gamma$ and $t' \in \Sigma\cap\Gamma^*$ imply that $x' \cdot t' \ge c > 0$, from the representation $\mathcal{K}_\Gamma(x') = i^n\Gamma(n) \int_{\Sigma\cap\Gamma^*} (x'\cdot t')^{-n} dt'$ it follows that $\mathcal{K}_\Gamma(x')$ is bounded by a constant.

3. Proof of the main theorem. In this section we shall prove Theorem 1. Let us introduce one more definition.

The real part of the Cauchy kernel

$$p_\Gamma(x,y) = \Re\mathcal{K}_\Gamma(x+iy) = \int_{\Gamma^*} e^{-y\cdot t}\cos x\cdot t dt, \quad x \in \mathbf{R}^n, \ \ y \in \operatorname{int}\Gamma \tag{10}$$

is called the *Poisson kernel* of Γ. (We use this name because the kernel (10) will play the same role as the Poisson kernel in $\mathbf{R}_+$, although usually ([**6**] or [**7**]) a somewhat different function is called the Poisson kernel of Γ.)

Let $\mathcal{P}_\Gamma$ denote the operator with kernel p_Γ

$$\mathcal{P}_\Gamma = \int_{\Gamma^*} F(x) p_\Gamma(x,y) dx, \quad y \in \operatorname{int}\Gamma$$

for functions F defined on Γ.

The proof of Theorem 1 is based on the following two theorems. Theorem A [**7**] is a generalization to n dimensions of Karamata's Tauberian theorem for the Laplace transform. Theorem B is an application of a theorem proved in [**4**] to the operator $\mathcal{P}_\Gamma$. We shall obtain Theorem 1 by the following simple argument. If R is regularly varying, then Theorem B implies that $\mathcal{P}_\Gamma R$ is regularly varying. Then, by an application of Parseval's equation we pass from $\mathcal{P}_\Gamma R(x)$ to $\mathcal{L}_\Gamma \widehat{R}(ix)$ (see Lemma

2). Finally, when we have the regular variation of the Laplace transform of $\widehat{R}$, the Tauberian Theorem A yields the regular variation of $\mathcal{I}_{G^*}\widehat{R}$.

THEOREM A [7]. *Let Γ be a cone in $\mathbf{R}^n$ and let F be a positive function defined on Γ. Let the Laplace transform $\mathcal{L}F(iy)$ be well defined for $y \in \operatorname{int}\Gamma^*$ and let*

$$\lim_{\lambda\to\infty}\frac{\mathcal{L}F(iy/\lambda)}{r(\lambda)} = \psi(y), \quad y \in \operatorname{int}\Gamma^*.$$

Then

$$\lim_{\lambda\to\infty}\frac{\mathcal{I}F(\lambda x)}{r(\lambda)} = \varphi(y), \quad x \in \operatorname{int}\Gamma,$$

uniformly in compact sets in $G\setminus\{0\}$ (where G is smaller than Γ),and $\mathcal{L}\varphi(iy) = \psi(y)\mathcal{K}_\Gamma(iy)$.

THEOREM B. *Let Γ be a cone in $\mathbf{R}^n$ and G a cone larger than Γ. Let $p_G(x,y)$ be the Poisson kernel of G. Let $0<\rho<n$. Then*

a) (i) $p_G(\lambda x,\lambda y) = \lambda^{-n}p_G(x,y), \; x\in\mathbf{R}^n, \quad y \in G$

(ii) $$\int_\Gamma \max(|x|^{-\eta-\rho},|x|^{-\eta-\rho})|p_\Gamma(x,y)|dx \le \mathcal{K}_G(iy), \quad y\in G$$

for some $\eta>0$.

b) *Consider the operator $\mathcal{P}_G$ with kernel p_G on $\Gamma\times G$*

$$\mathcal{P}_G F(y) = \int_\Gamma F(x)p_G(x,y)dx, \quad y \in G.$$

Let R be regularly varying at zero with index ρ in Γ. Then

$$\lim_{\lambda\to\infty}\frac{\mathcal{P}_G R(y/\lambda)}{r(\lambda)} = \mathcal{P}_G\varphi(y), \quad y\in\operatorname{int}G.$$

Proof. It was proved in [4] (Corollary 4 of Theorem 1) that for an operator whose kernel satisfies the conditions (i) and (ii) in a), the statement b) follows. Therefore, we only have to prove a). Since by definition $p_G(x,y) = \Re\mathcal{K}_G(x+iy)$ (i) follows by Lemma 1 (i).

To prove (ii) we split the integral into two parts

$$\int_\Gamma = \int_{|x|<1} + \int_{|x|>1} = I_1 + I_2$$

For the first integral we have

$$I_1 = \int_{|x|<1} |x|^{-\rho-\eta}|p_G(x,y)|dx < \mathcal{K}_G(iy)\int_{|x|<1}|x|^{-\rho-\eta}dx \tag{11}$$

by Lemma 1 (ii). And the last integral in (11) converges if $-\rho-\eta+n>0$, i.e. $\eta<n-\rho$.

For the second integral I_2 we have by Lemma 1 (iii)

$$I_2=\int\limits_{|x|>1}|x|^{-\rho+\eta}|p_G(x,y)|dx<\int\limits_{|x|>1}|x|^{-\rho+\eta}|\mathcal{K}_G(x)|dx \tag{12}$$

and by introducing the polar coordinates $r=|x|$ and $x'=x/|x|$ in the last integral we have

$$\begin{aligned}\int\limits_{|x|>1}|x|^{-\rho+\eta}|\mathcal{K}_G(x)|dx&=\int\limits_{\Sigma\cap\Gamma}dx'\int\limits_1^\infty r^{-\rho+\eta+n-1}|\mathcal{K}_G(rx')|dr\\&=\int\limits_{|x|>1}|\mathcal{K}_G(x')|dx'\int\limits_1^\infty r^{-\rho+\eta-1}dr\end{aligned} \tag{13}$$

since $\mathcal{K}_G$ is homogeneous of order $-n$, by Lemma 1 (i). Now by an application of Lemma 1 (v) we see that $\int\limits_{\Sigma\cap\Gamma}|\mathcal{K}_G(x')|dx<C$, and on the other hand $\int\limits_1^\infty r^{-\rho+\eta-1}dr$ converges if $-\rho+\eta<O$. Consequently, if we substitute (13) into (12) we see that I_2 is majorized by a constant, for $\eta<\rho$.

Now, if we choose $0<\eta<\min(\rho,n-\rho)$, formulae (12) and (11) together give (ii). This completes the proof of Theorem B.

The equality (14) in the following lemma (which is in fact Parseval's equality) will give an important step in the proof of Theorem 1.

LEMMA 2. *Let R be defined on a cone Γ and satisfy the conditions of Theorem 1. Let $\mathcal{P}_G$ be the Poisson operator for a cone G larger than Γ and let $\mathcal{L}$ be the Laplace operator. Then*

$$\mathcal{P}_GR(y)=\mathcal{L}\widehat{R}(iy),\quad y\in\operatorname{int}G. \tag{14}$$

Proof. We have

$$\mathcal{P}_GR(y)=\int\limits_\Gamma R(x)p_G(x,y)dx. \tag{15}$$

By definition, the Poisson kernel $p_G(x,y)=\int\limits_{G^*}e^{-y\cdot t}\cos x\cdot t dt$ is the cosine Fourier transform of the function $g_y(t)$ which is equal to $e^{-y\cdot t}$, for $t\in G^*$, and is equal to 0, for any other t. Thus we have from (15) (since R is supported in Γ)

$$\mathcal{P}_GR(y)=\int\limits_{\mathbf{R}^n}R(x)\widehat{g}_y(x)dx,\quad y\in\operatorname{int}G.$$

We now apply the Parseval equality to the last integral and obtain

$$\mathcal{P}_G R(y) = \int\limits_{\mathbf{R}^n} R(x)\widehat{g}_y(x)dx = \int\limits_{\mathbf{R}^n} \widehat{R}(t)g_y(t)dt = \int\limits_{G^*} \widehat{R}(t)e^{-y\cdot t}dt, \tag{16}$$

since g_y is supported in G^*. But the last integral in (16) is by definition

$$\int\limits_{G^*} \widehat{R}(t)e^{-y\cdot t}dt = \mathcal{L}\widehat{R}(iy), \quad y \in \operatorname{int} G.$$

From this and (16) equality (14) follows at once.

Proof of Theorem 1. We have by assumption

$$\lim_{\lambda\to\infty} \frac{R(x/\lambda)}{r(\lambda)} = \varphi(x), \quad x \in \Gamma$$

uniformly in compact sets in $\Gamma \setminus \{0\}$. An application of Theorem B b) yields

$$\lim_{\lambda\to\infty} \frac{\mathcal{P}_G R(y/\lambda)}{r(\lambda)} = \mathcal{P}_G\varphi(y), \quad y \in \operatorname{int} G \tag{17}$$

Now we apply Lemma 2, we substitute (14) (and a similar equality for φ) into (17) and obtain

$$\lim_{\lambda\to\infty} \frac{\mathcal{L}\widehat{R}(iy/\lambda)}{r(\lambda)} = \mathcal{L}\widehat{\varphi}(iy), \quad y \in \operatorname{int} G. \tag{18}$$

Next we shall apply Theorem A (the Tauberian theorem for the Laplace transform); then (18) implies

$$\lim_{\lambda\to\infty} \frac{\mathcal{I}_{G^*}\widehat{R}(\lambda x)}{r(\lambda)} = \Phi(x), \quad x \in G^*, \tag{19}$$

and uniformly in compact sets in $G_1^* \setminus \{0\}$ (where G_1^* is smaller than G^*). For the function Φ in (19) we have $\mathcal{L}\Phi(iy) = \mathcal{L}\widehat{\varphi}(iy)\mathcal{K}_{G^*}(iy)$, and since $\mathcal{K}_{G^*}(iy) = \mathcal{L}\theta_{G^*}(iy)$ we have

$$\mathcal{L}\Phi(iy) = \mathcal{L}\widehat{\varphi}(iy)\mathcal{L}\theta(iy) = \mathcal{L}(\widehat{\varphi} * \theta_{G^*})(iy).$$

From this by the uniqueness of the Laplace transform it follows that $\Phi(x) = \mathcal{I}_{G^*}\widehat{\varphi}(x)$, $x \in G$, and (19) becomes

$$\lim_{\lambda\to\infty} \frac{\mathcal{I}_{G^*}\widehat{R}(\lambda x)}{r(\lambda)} = \mathcal{I}_{G^*}\widehat{\varphi}(x), \quad x \in G^*.$$

Observe that when G is larger than Γ, then G^* is smaller than Γ^* and thus G_1^* is smaller than Γ^*. This completes the proof of the theorem.

4. Monotone Fourier transforms. So far we have proved that the regular variation of a function R in a cone Γ implies the regular variation of the primitive of its Fourier transform in a cone smaller than Γ^*. In this section we shall obtain the regular variation of $\widehat{R}$ itself, under the additional assumption that it is monotone. This will be obtained as a simple consequence of the following lemma.

The cone Γ defines a partial order in $\mathbf{R}^n$ in the following way: $x \le_\Gamma y$ if $y - x \in \Gamma$. A function $F : \Gamma \to \mathbf{R}$ is said to be *monotone increasing (decreasing)* in Γ if $x \le_\Gamma y$ implies $F(x) \le F(y)$ $(F(x) \ge F(y))$. For example, if F is positive on Γ, then $\mathcal{I}_\Gamma F(x)$ is monotone increasing in Γ.

LEMMA 3 [**7**]. *Let F be monotone on a cone Γ and let its primitive function be regularly varying on Γ, i.e.,*

$$\lim_{\lambda\to\infty} \frac{\mathcal{J}_\Gamma F(\lambda x)}{r(\lambda)} = \varphi(x), \quad x \in \Gamma.$$

Then

$$\lim_{\lambda\to\infty} \frac{F(\lambda x)}{\lambda^{-n} r(\lambda)} = \psi(x), \quad x \in \Gamma$$

where $\varphi(x) = \mathcal{I}_\Gamma \psi(x)$.

This lemma was proved in [**7**] (see the proof of Theorem 9.1). Now, the following corollary is obtained from Theorem 1 by a direct application of Lemma 3.

COROLLARY 1. *Let R satisfy the conditions of Theorem 1, and let moreover its Fourier transform $\widehat{R}$ be monotone decreasing on the cone G^*. Then (3) implies that*

$$\lim_{\lambda\to\infty} \frac{\widehat{R}(\lambda x)}{\lambda^{-n} r(\lambda)} = \widehat{\varphi}(x), \quad x \in G^*$$

uniformly in compact sets in $G^* \setminus \{0\}$.

5. Fourier transforms of monotone functions. The theorem we prove in this section is in a certain sense dual to Theorem 1: if a function is monotone and regularly varying at infinity in Γ, then its Fourier transform is regularly varying at zero in the dual cone Γ^*. Since the functions we deal with will not be integrable in the entire cone Γ, we introduce the principal value of the integral.

Let F be a locally integrable function on a cone Γ. The integral $\int\limits_\Gamma F(t)dt$ is said to converge in the sense of *principal value* if the limit $\lim\limits_{|b|\to\infty} \int\limits_{\Gamma\cap b-\Gamma} F(t)dt$ exists for $b \in \Gamma_1$ where Γ_1 is a cone smaller than Γ.

If F can be represented as $F(x) = \int\limits_{x+\Gamma} f(t)dt$, for some integrable function f, then F is called the primitive of f. Obviously, if $f \ge 0$, then F is monotone decreasing. In this case F is said to be *monotone primitive.*

THEOREM 2. *Let $0 < \rho < n$. Let Γ be a cone in $\mathbf{R}^n$. Let F be a monotone primitive function on Γ and let F be locally integrable on Γ. Then*

a) *the integral defining the Fourier transform*

$$\widehat{F}(x) = \int_{\Gamma} F(t) \cos x \cdot t dt, \quad x \in \Gamma^* \tag{20}$$

converges in the sense of principal value.

b) *If* $F(x) = \int_{x+\Gamma} f(t)dt$, *where* f *is monotone, and if* F *is regularly varying at infinity in* Γ *with index* $\rho - n$, *i.e. if*

$$\lim_{\lambda \to \infty} \frac{F(\lambda u)}{\lambda^{-n} r(\lambda)} = \varphi(u), \quad u \in \Gamma$$

uniformly in compact sets in $\Gamma \setminus \{0\}$ *then*

$$\left| \frac{\widehat{F}(x/\lambda)}{r(\lambda)} - \widehat{\varphi}(x) \right| \le \varepsilon(\lambda) |\mathcal{K}_{\Gamma^*}(x)|, \quad x \in \Gamma^*$$

and, consequently,

$$\lim_{\lambda \to \infty} \frac{\widehat{F}(x/\lambda)}{r(\lambda)} = \widehat{\varphi}(x), \quad x \in \Gamma$$

uniformly in compact sets in $\Gamma \setminus \{0\}$ *where* Γ_1 *is smaller than* Γ^*.

This theorem is a simple corollary of the following Theorem **C** which was proved in [**4**] (see Corollary 5 of Theorem 2).

We shall write $\langle a, b \rangle$ for the "interval" $a + \Gamma \cap b - \Gamma$.

THEOREM **C**. *Let* $0 < \rho < n$. *Let* Γ *and* G *be two cones in* $\mathbf{R}^n$. *Let* $k : \Gamma \times G \to \mathbf{R}$ *be such that* $k(\lambda u, x/\lambda) = k(u, x)$ *and* $\left| \int_{\langle a,b \rangle} k(u,x)du \right| \le C(x)$, *for some positive function* $C(x)$, $x \in G$ *and all* $\langle a, b \rangle \subset \Gamma$. *Let* F *be a function on* Γ *satisfying the conditions of Theorem 2. Then*

a) *the integral*

$$KF(x) = \int_{\Gamma} F(u) k(u,x) du, \quad x \in G$$

exists in the sense of principal value.

b)
$$\left| \frac{KF(x/\lambda)}{r(\lambda)} - K\varphi(x) \right| \le \varepsilon(\lambda) C(x), \quad x \in G.$$

where $e(\lambda) \to 0$, *as* $k \to \infty$.

We shall consider the following kernel $k(u,x) = \cos u \cdot x$ defined on $\Gamma \times \Gamma^*$ and we shall prove that it satisfies the conditions of Theorem **C**. This will prove that it is possible to apply Theorem **C** to the Fourier operator (20).

Now obviously, $\cos(u/\lambda \cdot \lambda x) = \cos u \cdot x$ and in the next lemma we prove that $k(u,x) = \cos u \cdot x$ satisfies also the second condition of Theorem **C** (with $C(x) = \mathcal{K}_{\Gamma^*}(x)$) and this will complete the proof of Theorem 2.

LEMMA 3. *Let Γ be a cone in $\mathbf{R}^n$ and let $\mathcal{K}_{\Gamma^*}$ be the Cauchy kernel of the dual cone Γ^*. Then*

$$\left| \int\limits_{\langle a,b \rangle} \cos x \cdot u du \right| \leq |\mathcal{K}_{\Gamma^*}(x)|, \quad x \in \Gamma^*$$

for all $\langle a,b \rangle \subset \Gamma$.

Proof. We shall prove the lemma by a procedure similar to the proof of Proposition 1. Denote by $\theta_{\langle a,b \rangle}$, the characteristic function of the interval $\langle a,b \rangle$. We shall show that for the Laplace transform of $\theta_{\langle a,b \rangle}$ the following representation holds

$$\mathcal{L}\theta_{\langle a,b \rangle}(z) = i^n \int\limits_{\Sigma \cap \Gamma} \frac{A(t')dt'}{(z \cdot t')^n}, \quad z \in T^{\Gamma^*}. \tag{21}$$

where $A(t')$ is a continuous positive function such that $A(t') \leq \Gamma(n)$.

If (21) holds, then by an argument similar to the one used in the proof of Proposition 1, both sides of (21) are analytic functions, and they have an analytic continuation in a domain containing the real cone $\operatorname{int} \Gamma^*$, so that by putting $z = x \in \operatorname{int} \Gamma^*$ in (21) we have

$$\mathcal{L}\theta_{\langle a,b \rangle}(x) = i^n \int\limits_{\Sigma \cap \Gamma} \frac{A(t')dt'}{(x \cdot t')^n}$$

and from this it follows that

$$|\mathcal{L}\theta_{\langle a,b \rangle}(x)| \leq \Gamma(n) \int\limits_{\Sigma \cap \Gamma} \frac{dt'}{(x \cdot t')^n}, \quad x \in \operatorname{int} \Gamma^*. \tag{22}$$

Now, the last integral in (22) equals $|\mathcal{K}_{\Gamma^*}(x)|$ (see (9)) and $\mathcal{L}\theta_{\langle a,b \rangle}(x) = \int\limits_{\langle a,b \rangle} e^{ix \cdot u} du$, so that (22) becomes

$$\left| \int\limits_{\langle a,b \rangle} e^{ix \cdot u} du \right| \leq |\mathcal{K}_{\Gamma^*}(x)|, \quad x \in \operatorname{int} \Gamma^*$$

and from this the lemma follows immediately.

Thus we only have to prove (21). Since both sides in (21) are analytic functions in T^{Γ^*}, it is sufficient to prove this equality for $z = iy$.

$$\mathcal{L}\theta_{\langle a,b \rangle}(iy) = \int\limits_{\langle a,b \rangle} e^{-y \cdot t} dt = \int\limits_{\Sigma \cap \Gamma} dt' \int\limits_{\alpha(t')}^{\beta(t')} e^{-ry \cdot t'} r^{n-1} dr,$$

where $\alpha(t')$ is (the distance from 0 to) the intersection of the ray rt' with the cone $a + \Gamma$, and $\beta(t')$ is the intersection of rt' with the cone $b - \Gamma$.

Now, introducing the change of variables $r(y \cdot t') = u$ in the last integral, and then introducing the notation

$$A(t') = \int\limits_{\beta(t')(y\cdot t')}^{\alpha(t')(y\cdot t')} e^{-u}u^{n-1}du$$

(obviously, $A(t') \leq \int_0 e^{-u}u^{n-1}du = \Gamma(n)$) we have

$$\mathcal{L}\theta_{\langle a,b\rangle}(iy) = \int\limits_{\Sigma\cap\Gamma} \frac{A(t')dt'}{(y\cdot t')^n} = i^n \int\limits_{\Sigma\cap\Gamma} \frac{A(t')dt'}{(iy\cdot t')^n},$$

which proves (21), and this completes the proof of the lemma.

At the end let us remark that it seems that, by a better technique (for example, a better definition of regularly varying functions) it should be possible to obtain the behaviour of the Fourier transform (in Theorem 1) in the whole dual cone, and not only in a cone which is smaller.

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