

ORIENTATION OF ABSOLUTE SPACE S^n

Vladimir Janković

Abstract. We consider one way of founding the orientation of absolute space S^n . Briefly, we show how the orientation can be introduced by using the first two groups of axioms: axioms of incidence and axioms of order. In [1] the orientation of E^2 is found in a similar way, but using an analytical method which needs all five groups of axioms.

1. Oriented Simplexes and Chaiias

An oriented simplex is a simplex whose vertices are ordered. We shall consider only oriented simplexes and for the sake of simplicity we shall call them simplexes. By the simplex $A_0A_1\dots A_n$ we shall mean the simplex whose first vertex is A_0 , whose second vertex is A_1 and so on.

Simplexes $A_0A_1\dots A^n$ and $B_0B_1\dots B_n$ are connected if $A_i = B_{i-1}$, $i = 1, \dots, n$. A chain is a finite sequence of simplexes such that any two consecutive members are connected. A chain is closed if its initial and terminal members coincide. By the chain $A_0A_1\dots A^n$, ($m > n$) we shall mean the sequence of simplexes $A_iA_{i+1}\dots A_{i+n}$, $i = 0, 1, \dots, m - n$. By the closed chain $A_0A_1\dots A^n$ ($m > n$) we shall mean the sequence of simplexes $A_iA_{i+1}\dots A_{i+n}$ $i = 0, 1, \dots, m$, where $A_{m+j} = A_j$ for $j = 0, 1, \dots, n$. Chain C connects simplex S with simplex S' if it starts at S and if it terminates at S' .

THEOREM 1. *For any two simplexes S and S' there exists a chain which connects S with S' .*

Proof. Let $S = A_0A_1\dots A^n$ and $S' = B_0B_1\dots B^n$. Furthermore,

- let C_1 be a point which doesn't lie on the hyperplane $A_0A_1\dots A^n$, and which is distinct from the point B_0 ;

- let C_2 be a point which doesn't lie on the hyperplane $A_1A_2\dots A^nC_1$, and which doesn't lie on the lines C_1B_0 and B_0B_1 ;

- let C_3 be a point which doesn't lie on the hyperplane $A_3A_4\dots A^nC_1C_2$, and which doesn't lie on the planes $C_1C_2B_0$, $C_2B_0B_1$ and $B_0B_1B_2$;...

- let C_n be a point which doesn't lie on the hyperplanes $A_n C_1 \dots C_{n-1}$, $C_1 C_2 \dots, C_{n-1} B_0, \dots, C_{n-1} B_0 \dots B_{n-2}$ and $B_0 B_1 \dots B_{n-1}$. It is obvious that the chain $C = A_0 A_1 \dots A_n C_1 \dots C_n B_0 B_1 \dots B_n$ connects the simplex S with the simplex S' . ■

2. Parity of Chains

A couple of connected simplexes $A_0 A_1 \dots A_n$ and $A_1 \dots A_n A_{n+1}$ is anti-oriented if the vertices A_0 and A_{n+1} lie on the same side of the hyperplane determined by the common side $A_1 \dots A_n$ for n odd, and if the vertices A_0 and A_{n+1} lie on the opposite sides of the hyperplane determined by the common side $A_1 \dots A_n$ for n even. A parity of a chain is the parity of the number of anti-oriented couples of consecutive members of that chain.

THEOREM 2. *Closed chains are even.*

Proof for $n = 1$. Suppose that the closed chain $A_0 A_1 \dots A_{m-1}$ is given. The points A_0, A_1, \dots, A_{m-1} can be enumerated by the integers a_0, a_1, \dots, a_{m-1} such that $(a_i - a_j)(a_j - a_k) > 0$ if and only if A_j lies between A_i and A_k . The couple of segments $A_i A_{i+1}$ and $A_{i+1} A_{i+2}$ is anti-oriented if and only if $(a_i - a_{i+1})(a_{i+1} - a_{i+2}) < 0$. Since

$$\prod_{i=1}^{m-1} (a_i - a_{i+1})(a_{i+1} - a_{i+2}) = \prod_{i=1}^{m-1} (a_i - a_{i+1})^2 > 0,$$

the number of anti-oriented couples of consecutive members of the given chain is even, i.e. the given chain is even. ■

Let H be a hyperplane and let A and B be two points which don't lie on H . Let us define $a(A, H, B)$ and $b(A, H, B)$ as

$$a(A, H, B) = \begin{cases} 1, & A, B \overset{\cdot\cdot}{\div} H \\ -1, & A, B \overset{\cdot}{\div} H \end{cases}$$

$$b(A, H, B) = \begin{cases} 1, & A, B \overset{\cdot}{\div} H \\ -1, & A, B \overset{\cdot\cdot}{\div} H \end{cases}$$

These two functions have the following two properties:

a) If points A , B and C don't lie on the hyperplane H , then

$$a(A, H, B)a(B, H, C)a(C, H, A) = 1,$$

$$b(A, H, B)b(B, H, C)b(C, H, A) = -1.$$

b) If three points A , B and C and the plane P of codimension 2 determine three distinct hyperplanes, then

$$a(A, BP, C)a(B, CP, A)a(C, AP, B) = -1,$$

$$b(A, BP, C)b(B, CP, A)b(C, AP, B) = 1.$$

LEMMA. Let A_0, A_1, \dots, A_m be points in S^n ($n > 1$). Then there exist points A'_0, A'_1, \dots, A'_m in S^n such that no $n + 1$ of them lie on one hyperplane and such that

$$B(A'_i, A'_{k_1}, A'_{k_2}, \dots, A'_{k_n}, A'_j) \Leftrightarrow B(A_i, A_{k_1}, A_{k_2}, \dots, A_{k_n}, A_j) \quad (*)$$

provided the points, $A_{k_1}, A_{k_2}, \dots, A_{k_n}$ determine a unique hyperplane and the points A_i and A_j don't lie on that hyperplane.

Proof. Let l_0 be a line passing through A_0 which doesn't lie on any hyperplane determined by n points among A_0, A_1, \dots, A_m . Let A'_0 be a point lying on l_0 such that the segment $]A_0, A'_0]$ doesn't have a common point with any considered hyperplane. The points A'_0, A_1, \dots, A_m satisfy the given condition. If $i = 0$, the condition (*) is satisfied because, according to the property a) of the function b ,

$$b(A'_0, A_{k_1} A_{k_2} \dots A_{k_n}, A_j) = -b(A'_0, A_{k_1} A_{k_2} \dots A_{k_n}, A_0) b(A_0, A_{k_1} A_{k_2} \dots A_{k_n}, A_j)$$

and, according to the way point A'_0 is chosen,

$$b(A'_0, A_{k_1} A_{k_2} \dots A_{k_n}, A_0) = -1.$$

The case $j = 0$ can be considered in the same way. If $k_1 = 0$, the condition (*) is satisfied, because, according to the property b) of the function b ,

$$\begin{aligned} b(A_i, A'_0 A_{k_2} \dots A_{k_n}, A_j) &= b(A'_0, A_i A_{k_2} \dots A_{k_n}, A_j) b(A_i, A_j A_{k_2} \dots A_{k_n}, A'_0), \\ b(A_i A_0 A_{k_2} \dots A_{k_n}, A_j) &= b(A_0, A_i A_{k_2} \dots A_{k_n}, A_j) b(A_i, A_j A_{k_2} \dots A_{k_n}, A_0), \end{aligned}$$

and the equalities

$$\begin{aligned} b(A'_0, A_i A_{k_2} \dots A_{k_n}, A_j) &= b(A_0, A_i A_{k_2} \dots A_{k_n}, A_j), \\ b(A_i, A_j A_{k_2} \dots A_{k_n}, A'_0) &= b(A_i, A_j A_{k_2} \dots A_{k_n}, A_0), \end{aligned}$$

have been already proved. The cases $k_2 = 0, \dots, k_n = 0$ can be considered in the same way.

Repeating the same procedure with the points A_1, \dots, A_m , we shall get the points A'_0, A'_1, \dots, A'_m which satisfy all of the needed conditions. ■

Proof of Theorem 2 for odd $n > 1$. The closed chain $A_0 A_1 \dots A_{m-1}$ is even if and only if -1 occurs an even number of times among the numbers $b(A_i, A_{i+1} A_{i+2} \dots A_{i+n}, A_{i+n+1})$, $i = 0, 1, \dots, m-1$, i.e., if and only if

$$\prod_{i=0}^{m-1} b(A_i, A_{i+1} A_{i+2} \dots A_{i+n}, A_{i+n+1}) = 1.$$

We shall prove our statement by induction on m .

For $m = n + 1$ the statement is valid, because all of the numbers $b(A_i, A_{i+1} A_{i+2} \dots A_{i+n}, A_{i+n+1})$ are equal to -1 and m is even.

Let us suppose that the statement is valid for integer $m > n$. Suppose that the closed chain $A_0 A_1 \dots A_m$, is given. By the previous lemma, we can suppose

that no $n + 1$ among the points A_0, A_1, \dots, A_m belong to one hyperplane. By the induction hypothesis, the statement is valid for the closed chain $A_0 A_1 \dots, A_{m-1}$, and therefore $P = 1$, where

$$P = \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot b(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_0) \cdot \prod_{j=0}^{n-1} b(A_{m-n+j}, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_{j+1}).$$

It remains to show that $Q = 1$, where

$$Q = \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot b(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_m) \cdot \left(\prod_{j=0}^{n-1} b(A_{m-n+j}, A_{m-n+j+1} \dots A_m A_0 \dots A_{j-1}, A_j) \right) \cdot b(A_m, A_0 \dots A_{n-1}, A_n).$$

Using the properties b) and a) of the function b we get

$$\begin{aligned} Q &= \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot b(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_m) \cdot \prod_{j=0}^{n-1} [b(A_m, A_{m-n+j} \dots A_{m-1} A_0 \dots A_{j-1}, A_j) \cdot b(A_{m-n+j}, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_m)] \cdot b(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_m), \\ Q &= \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot b(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_m) \cdot b(A_m, A_{m-n} \dots A_{m-1} A_0) \cdot \prod_{j=0}^{n-1} [b(A_m, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_{j+1}) \cdot b(A_{m-n+j}, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_m)], \\ Q &= \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot (-b(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_0)) \cdot \prod_{j=0}^{n-1} (-b(A_{m-n+j}, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_{j+1})), \\ Q &= \left(\prod_{i=0}^{m-n-2} b(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot b(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_0) \cdot \prod_{j=0}^{n-1} b(A_{m-n+j}, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_{j+1}), \end{aligned}$$

$$Q = P = 1. \blacksquare$$

Proof of Theorem 2 for n even. The closed chain $A_0 A_1 \dots A_{m-1}$ is even if and only if -1 occurs an even number of times among the numbers $a(A_i, A_{i+1} A_{i+2} \dots A_{i+n}, A_{i+n+1})$, $i = 0, 1, \dots, m-1$, i.e., if and only if

$$\prod_{i=0}^{m-1} a(A_i, A_{i+1} A_{i+2} \dots A_{i+n}, A_{i+n+1}) = 1.$$

We shall prove our statement by induction on m .

For $m = n + 1$ the statement is valid, because all of the numbers $a(A_i, A_{i+1} A_{i+2} \dots A_{i+n}, A_{i+n+1})$ are equal to 1.

Let us suppose that the statement is valid for integer $m > n$. Suppose that the closed chain $A_0 A_1 \dots A_m$ is given. By the previous lemma, we can suppose that no $n + 1$ among the points A_0, A_1, \dots, A_m belong to one hyperplane. By the induction hypothesis, the statement is valid for the closed chain $A_0 A_1 \dots A_{m-1}$, and therefore $P = 1$, where

$$P = \left(\prod_{i=0}^{m-n-2} a(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot a(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_0) \cdot \prod_{j=0}^{n-1} a(A_{m-n+j}, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_{j+1}).$$

It remains to show that $Q = 1$, where

$$Q = \left(\prod_{i=0}^{m-n-2} a(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot a(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_0) \cdot \left(\prod_{j=0}^{n-1} a(A_{m-n+j}, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_{j+1}) \right) \cdot a(A_m, A_0 \dots A_{n-1}, A_n).$$

Using the properties b) and a) of the function a we get

$$Q = \left(\prod_{i=0}^{m-n-2} a(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot a(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_m) \cdot \left(\prod_{j=0}^{n-1} [-a(A_m, A_{m-n+j} \dots A_{m-1} A_0 \dots A_{j-1}, A_j) \cdot a(A_{m-n+j}, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_m)] \right) \cdot a(A_m, A_0 \dots A_{n-1}, A_n),$$

$$Q = \left(\prod_{i=0}^{m-n-2} a(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot a(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_m) \cdot a(A_m, A_{m-n} \dots A_{m-1} A_0) \cdot \prod_{j=0}^{n-1} [a(A_m, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_{j+1}) \cdot a(A_{m-n+j}, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_m)],$$

$$Q = \left(\prod_{i=0}^{m-n-2} a(A_i, A_{i+1} \dots A_{i+n}, A_{i+n+1}) \right) \cdot a(A_{m-n-1}, A_{m-n} \dots A_{m-1}, A_0) \cdot \prod_{j=0}^{n-1} a(A_{m-n+j}, A_{m-n+j+1} \dots A_{m-1} A_0 \dots A_j, A_{j+1}),$$

$$Q = P = 1. \blacksquare$$

THEOREM 3. *If the chains C and C' have the same origin and the same end, they have the same parity.*

Proof. Let C'' be a chain which connects the common end of C and C' with their common origin. If we extend C by C'' we get a closed chain. We thus conclude that the total number of anti-oriented couples of consecutive members of C and C'' is even. Therefore C and C'' have the same parity. In the same way we conclude that C' and C'' have the same parity. It follows that C and C' have the same parity. \blacksquare

3. Orientations

Simplex S has the same orientation as simplex S' , or briefly $S \rightrightarrows S'$, if each chain which connects them is even. Simplex S has the opposite orientation to simplex S' , or briefly $S \leftrightharpoons S'$, if each chain which connects them is odd.

THEOREM 4. *Relation \rightrightarrows is an equivalence relation which defines the partition of the family of simplexes into two equivalence classes.*

Proof. This relation is reflexive because each closed chain is even.

Let $S \rightrightarrows S'$ and $S' \rightrightarrows S''$. Let C be a chain which connects S' with S'' , and let C'' be the chain which is the extension of C by C' . The chain C'' is even, because chains C and C' are even. Therefore $S \rightrightarrows S''$. We conclude that the relation \rightrightarrows is transitive.

Let $S \rightrightarrows S'$. Let C be a chain which connects S with S' , let C' be a chain which connects S' with S , and let C'' be the chain which is the extension of C by C' . Chain C'' is even, because chains C and C' are even. Therefore $S' \rightrightarrows S$. We conclude that the relation \rightrightarrows is symmetric.

Let $S \leftrightharpoons S'$ and $S' \leftrightharpoons S$. Let C be a chain which connects S with S' , let C' be a chain which connects S' with S , and let C'' be the extension of C by C' . The chain C'' is even, because chains C and C' are odd. Therefore $S \rightrightarrows S''$. \blacksquare

An orientation of n -dimensional absolute space S^n is any equivalence class with respect to the relation \rightrightarrows . There are two opposite orientations of space S^n .

THEOREM 5. *Let p be a permutation of numbers $0, 1, \dots, n$. Simplex $A_{p_0} A_{p_1} \dots A_{p_n}$ has the same orientation as simplex $A_0 A_1 \dots A_n$ if and only if the permutation p is even.*

Proof. First, let us prove that the statement is valid for the transposition $(0,1)$. Let A be any inner point of the $n-1$ -dimensional simplex $A_1 \dots A_n$. Let us consider the chain $A_1 A_0 A_2 \dots A_n A A_0 A_1 \dots A_n$ which connects the simplex $A_1 A_0 A_2 \dots A_n$ with the simplex $A_0 A_1 \dots A_n$. The number of anti-oriented couples of consecutive members of this chain is 3 if n is odd, and it is $n-1$ if n is even. Therefore, this chain is odd, and so the simplex $A_1 A_0 A_2 \dots A_n$ has the opposite orientation to the simplex $A_0 A_1 \dots A_n$.

Now, let us prove that the statement is valid for the cyclic permutation $(0, 1, \dots, n)$. It has the same parity as n . Let us consider the chain $A_1 A_2 \dots A_n A_0 A_1 \dots A_n$ which connects the simplex $A_1 \dots A_n A_0$ with the simplex $A_0 A_1 \dots A_n$. The number of anti-oriented couples of consecutive members of this chain is 0 if n is even, and it is n for n odd. Therefore, this chain has the same parity as n . So, the simplex $A_1 \dots A_n A_0$ has the same orientation as the simplex $A_0 A_1 \dots A_n$ if and only if the cyclic permutation $(0, 1, \dots, n)$ is even.

Finally, let us prove that if the statement is valid for permutations p and q , then it is valid for their composition qp . Let S be an arbitrary simplex, let S' be the simplex obtained from S by reordering its vertices by the permutation p , and let S'' be the simplex obtained from S' by reordering its vertices by the permutation q . Simplex S'' arises from simplex S by reordering its vertices by the permutation qp . We shall consider four cases:

1. If the permutations p and q are even, the permutation qp is even. From $S \rightrightarrows S'$ and $S' \rightrightarrows S''$ it follows that $S \rightrightarrows S''$.
2. If the permutation p is even and if the permutation q is odd the permutation qp is odd. From $S \rightrightarrows S'$ and $S' \leftrightharpoons S''$ it follows that $S \leftrightharpoons S''$.
3. If the permutation p is odd and if the permutation q is even, the permutation qp is odd. From $S \leftrightharpoons S'$ and $S' \rightrightarrows S''$ it follows that $S \leftrightharpoons S''$.
4. If the permutations p and q are odd, the permutation qp is even. From $S \leftrightharpoons S'$ and $S' \leftrightharpoons S''$ it follows that $S \rightrightarrows S''$.

It remains to remark that the transposition $(0,1)$ and the cyclic permutation $(0, 1, \dots, n)$ generate all permutations of numbers $0, 1, \dots, n$. ■

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Odsek za matematiku
Prirodno-matematički fakultet
Studentski trg 16
11000 Beograd

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