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TWO EXAMPLES OF Q-TOPOLOGIES

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Abstract. A pair (Y, τ), where Y is an internal set, whereas τ is a topology (usually external) on Y, is called a *-topological space if τ has an internal base. The main example is $(^*X, \overline{\tau})$ where (X, τ) is a standard topological space and $\overline{\tau}$ the topology generated by $^*\tau$. This is the so called Q-topology on $^*X$ induced by (X, τ), a notion introduced by A. Robinson in [**4**]. This note contains negative answers to some questions of R. W. Button, [**1**], who asked whether the following implications

$$(^*X, \overline{\tau}) \text{ normal } \Rightarrow (X, \tau) \text{ normal}$$
$$(X, \tau) \text{ scattered } \Rightarrow (^*X, \overline{\tau}) \text{ scattered}$$

hold in some enlargement.

1. The first question

THEOREM 1.1. *Let us assume that the nonstandard model $^*\mathcal{M}$ has the property* $\operatorname{cof}(^*\omega) = |^*(2^\omega)|$; *in other words the external cofinality of $^*\omega$ is equal to the external cardinality of $^*(2^\omega)$. Then there exists a counterexample for the first question.*

It seems natural to conjecture that every non-standard model contains a counterexample for the first question.

To prove the theorem one should start with a non-normal space (X, τ). A standard example is the Niemytzki plane (see R. Engelking [**2**]). The proof could be carried on for the Niemytzki plane but we shall use another (similar) example which is easier to handle.

Let T be the tree of finite 0,1 – sequences and B a family of maximal branches in this tree of the maximal cardinality, i.e., $|B| = c$. The space (X, τ) is defined as follows: $X = T \cup B$, T is a set of isolated points and a typical neighborhood of $d \in B$, $d : \omega \to 2$, is $N_{d,m} = \{d\} \cup \{d|\, n : n \geq m\}$, where $a|\, n$ denotes the restriction of a on the natural number n.

Recall the well-known argument that this space is not normal. (X, τ) is separable, T being the dense set, consequently there exists at most c continuous

AMS Subject Classification (1980): Primary 03M05, 54J05 functions on X. Observing that B is a closed, discrete set of cardinality $c = 2^{\aleph_0}$ we see that any function $f : B \to R$ is continuous. If (X, τ) were normal, then by Titze-Urisohn theorem f can be extended to the whole set X which is a contradiction, for there would exist at least 2^c different continuous functions.

Let us prove now that $({}^*X, \overline{\tau})$ is not only normal but also paracompact in a very strong way.

Looking from outside ${}^*X = {}^*T \cup {}^*B$ is a pseudo-tree (non-wellordered) with set ${}^*B$ of maximal branches. The topology $\overline{\tau}$ on ${}^*X$ is described as follows. ${}^*T$ is a set of isolated points while a typical neighborhood of $d \in {}^*B$, $d : {}^*\omega \to 2$ is ${}^*N_{d,m} = \{d\} \cup \{a| \, n : n \geq m\}$ where m is a nonstandard natural number.

A simple consequence of our cardinality assumption, $\operatorname{cof}({}^*\omega) = k = |{}^*(2^\omega)|$, is that any intersection of less than k open sets in $({}^*X, \overline{\tau})$ is an open set.

Remark. Let us note that a model ${}^*\mathcal{M}$ with the property $\operatorname{cof}({}^*2) = |{}^*(2^\omega)|$ can be constructed under various set theoretical assumptions. For example the existence of $2^{\aleph_0}$-scale (see [**3**, p. 260]) implies this equality if the nonstandard model is a D-ultrapower where E is any nontrivial ultrafilter on ω. Recall that a $2^{\aleph_0}$-scale exists under CH or $MA + \neg CH$. On the other hand under GCH one can get a nonstandard model ${}^*\mathcal{M}$ which is $|{}^*\mathcal{M}|$-saturated which implies the equality $\operatorname{cof}({}^*\omega) = {}^*(2^\omega) = |{}^*\mathcal{M}|$. Indeed, if $A \subset {}^*\omega$ and $|A| < |{}^*\mathcal{M}|$ then $\mathcal{A} = \{[m, \infty) : m \in A\}$ is a family of internal sets of cardiiiality less than $|{}^*\mathcal{M}|$ which means that $\bigcap \mathcal{A} \neq \infty$ and A is not cofinal in ${}^*\omega$.

To finish the proof of the theorem it is enough to prove the following lemma.

LEMMA. *Let (Y, τ) be a nulldimensional topological space such that $|Y| = k$ and the intersection of $< k$ open sets is again an open set. Then every open cover of Y contains a disjoint open refinement. In particular, this space is paracompact (or strongly paracompact).*

Proof of Lemma. We can assume that the cardinality of the cover $\mathcal{U}$ is $\leq k$ and that its members are clopen. If $\{V_\alpha | \, \alpha < \lambda\}$ is a well-ordering of $\mathcal{U}$, $\lambda \leq k$, then a required refinement is defined by $U_\alpha = V_\alpha \setminus \bigcup\{U_\beta | \, \beta < \alpha\}$, $\alpha < \lambda$. This completes the proof of the theorem.

The example above also serves to provide the following consequence.

COROLLARY 2.1. *The implication $({}^*X, \tau)$ paracompact $\Rightarrow$ (X, τ) paracompact cannot unconditionally hold in any nonstandard model.*

2. The second question

Recall that a topological space (Y, τ) is called scattered if each nonempty subspace $A \subset Y$ contams an isolated point. Iterating the Cantor-Bendixon derivative it is possible to define an ordinal-valued function $r : Y \to \alpha$ such that $r^{-1}(0) = \operatorname{ip}(Y)$, where $\operatorname{ip}(A)$ denotes the set of isolated points in A, and $r^{-1}(\beta) = \operatorname{ip}(Y \setminus \bigcup\{A_\gamma | \, \gamma < \beta\})$, $\beta < \alpha$, whereas $Y = \bigcup_{\beta<\alpha} r^{-1}(\beta)$. The function r^{-1} will be called the rank function of the space Y.

THEOREM 2. *Get* (X, τ) *be a scattered topological space and r the corresponding rank function. Then* $({}^*X, \overline{\tau})$ *is scattered if and only if* $\sup r(X)$ *is a finite ordinal.*

Proof. If $\sup r(X)$ is finite, then ${}^*r$ is the rank function of $({}^*X, \overline{\tau})$, which proves that the last space is scattered. Let us assume that $r(X)$ contains infinite ordinals. By the definition of the rank function r one has

$$(\forall x \in X)(\forall O \text{ open set})(x \in O \Rightarrow r(x) \leq \sup r(O \setminus \{x\}) + 1),$$

since otherwise the rank of x would be smaller. Let $A = \{x \in^* X \mid {}^*r(x) \text{ is infinite}\}$. By the Transfer Principle one has that every internal neighborhood of a point $x \in A$ contains a point $y \neq x$ of infinite rank, $y \in A$, which means that A has no isolated points because internal neighborhoods make a base of the Q-topology. Hence, $({}^*X, \overline{\tau})$ is not scattered. Let us note that A is actually perfect, because the function ${}^*r : {}^*X \to {}^*r(X)$ is upper semicontinuous.

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