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ON CHARACTERIZATIONS OF INNER-PRODUCT SPACES

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Abstract. The generalized inner-product (x, y) in a normed linear space X is the right Gateaux derivative of the functional $\|x\|^2/2$ at x in the direction of y. The orthogonality relation for the generalized inner-product is $x \perp_G y \Leftrightarrow (x, y) = 0$. Tapia has proved that X must be an inner-product space if the generalized inner-product is either symmetric or linear in y, and Detlef Laugwitz showed that if dimension $X \geq 3$ and the orthogonality for generalized inner-product is symmetric, then X is an inner-product space. In this note we discuss this orthogonality relation and provide alternative proofs of the results of Tapia and Laugwitz.

Let X be a real normed space and let $g(x) = \|x\|$ be the norm functional. $\acute{q}_+(x, y)(\acute{q}_-(x, y))$ is the right (left) Gateaux derivative of q at x in the direction of y. The right Gateaux derivative of the functional $x \to g^2(x)/2$ at x in the direction of y is called the generalized inner-product of x with y and is denoted by (x, y). We will say x is G-orthogonal to y ($x \perp_G y$) if $(x, y) = 0$. Since $(x, y) = \|x\|\ \acute{q}_+(x, x)$, $x \perp_G y \Leftrightarrow$ either $x = 0$ or $\acute{q}_+(x, y) = 0$.

The following lemma collects some of the well-known properties of the Gateaux derivatives of the norm.

LEMMA 1. *Let $x \neq 0$, $y, z \in X$ and a and $b \geq 0$ be numbers. Then*

(i) $\acute{q}_+(x, y + z) \leq \acute{q}_+(x, y) + \acute{q}_+(x, z)$.

(ii) $\acute{q}_+(x, by) = b\acute{q}_+(x, y)$.

(iii) $\acute{q}_+(ax, y) = \acute{q}_+(x, y)$, *for* $a > 0$;
$= -\acute{q}_-(x, y)$, *for* $a < 0$.

(iv) $-\acute{q}_+(x, -y) = \acute{q}_-(x, y) \leq \acute{q}_+(x, y)$.

(v) $\acute{q}_+(x, \cdot)$ *is a linear functional if and only if* $\acute{q}_+(x, \cdot) = \acute{q}_-(x, \cdot)$.

(vi) $|\acute{q}_+(x, y)| \leq \|y\|$.

(vii) $\acute{q}_+(x, ax + by) = a\|x\| + b\acute{q}_+(x, y)$.

Proof. See James [**3**, page 272].

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Let us recall the notion of orthogonality in a normed linear space suggested by Birkhoff [**1**] and discussed by James [**3**]. We say x is J-orthogonal to y ($x \perp_J y$) if $\|x + kx\| \geq \|x\|$ for all real k. Some of the useful facts about J-orthogonality are given in the following:

LEMMA 2. (i) *$x \perp_J y \Rightarrow ax \perp_J by$ for all a and b.*

(ii) *For $0 \neq x$ and $y \in X$, there exist numbers a and b such that $x \perp_J ax + y$ and $bx + y \perp_J x$.*

(iii) *The number a (respectively b) in* (ii) *is unique if and only if the space X is smooth (respectively strictly convex).*

(iv) *$x \perp_J y$ if and only if $\acute{q}_+(x, y) \geq 0$ and $\acute{q}_+(x, -y) \geq 0$.*

Proof. See James [**3**].

THEOREM 1. *If x and y are linearly independent elements of X, then there exists a unique number b such that $x \perp_G bx + y$.*

Proof. Take $b = -\acute{q}_+(x, y)/\|x\|$. Then $\acute{q}_+(x, bx + y) = b\|x\| + \acute{Q}_+(x, y) = 0$. Thus $x \perp_G bx + y$. The uniqueness of b also follows.

For G-orthogonality there may be no number b such that $bx + y \perp_G x$, as the following example shows.

Example 1. Consider R^2 with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$. Let $x = (1, 0)$ and $y = (0, 1)$. We have

$$\begin{aligned} \acute{q}_+(sx + y, x) &= \lim_{t \to 0_+} (\|(s + t, 1)\| - \|(s, 1)\|)/t \\ &= \lim_{t \to 0_+} (|s + t| - |s|)/t = 1, \text{ for } s \geq 0; \\ &\qquad\qquad\qquad\qquad = -1, \text{ for } s < 0. \end{aligned}$$

Thus $\acute{q}_+(sx + y, x) \neq 0$ for all s.

THEOREM 2. (i) *X is smooth if and only if $x, y \in X$ and $x \perp_G y \Rightarrow x \perp_G y$.* (ii) *$X$ is strictly convex if and only if $\alpha x + y \perp_G x$ and $\beta x + y \perp_G x \Rightarrow \alpha = \beta$.*

Proof. If X is smooth, then $x \perp_J y$ if and only if the Gateaux derivative of the norm at x in the direction of y is zero. The orthogonalities are the same.

If X is not smoth, then there exist $0 \neq x$ and y such that $x \perp_J y$ and $x \perp_J x + y$. Then $x \perp_G y$ and $x \perp_G x + y$. But that means $\acute{q}_+(x, y) = 0$ and $\acute{q}_+(x, x + y) = 0 = \|x\| + \acute{q}_+(x, y)$, which is false.

(ii) If X is strictly convex and $\alpha x + y \perp_G x$, $\beta x + y \perp_G x$, then $\alpha x + y \perp_J x$, $\beta x + y \perp_J x$ and therefore $\alpha = \beta$. If X is not strictly convex, then choose z and y such that $\|z\| = \|y\| = \|sz + (1 - t)y\| = 1$ for $0 \leq t \leq 1$. For $0 < s < 1$

$$\acute{q}_+(s(z - y) + y, z - y) = \lim_{t \to 0_+} (\|(s + t)(z - y) + y\| - \|s(z - y) + y\|)/t = 0.$$

Thus $sx + y \perp_G x$ for $0 < s < 1$ where $x = z - y$. That completes the proof of the theorem.

The following result is Theorem 3.5 of James [**3**]. In view of the results above, we are able to give a shorter proof of it.

THEOREM 3. *Iff in a normed linear space X, the G-orthogonality is symmetric ($x \perp_G y \Rightarrow y \perp_G x$), then the J-orthogonality is also symmetric and X is both strictly convex and smoth.*

Proof. Suppose x and y are linear idependent elements of X such that $\alpha x + y \perp_G x$ and $\beta x + y \perp_G x$. Then the symmetry of G-orthogonality $\alpha = \beta = -\acute{q}_+(x,y)/\|x\|$. Therefore X is strictly convex.

Suppose X is not smooth. Then there exist $x, y \in X$ such that $x \perp_J y$ but not $x \perp_G y$. Chose $b \neq 0$ such that $y \perp_G by + x$. Then $by + x \perp_G y$. Since G-orthogonality implies J-orthogonality therefore $by + x1 \perp_J y$ which contradicts the strict convexity of the space. Hence X is smooth and both of the orthogonalities are the same. That gives the result.

COROLLARY 1. (Laugwitz [**4**, Theorem 4)). *Let X be a normed linear space of dimension ≥ 3. Then X is an inner-product space if and only if $(x,y) = 0$ implies $(y,x) = 0$.*

Proof. If X is an inner product space, then the generalized inner-product is the inner-product and therefore $(x,y) = 0 \Rightarrow (y,x) = 0$.

If $(x,y) = 0 \Rightarrow (y,x) = 0$, then by Theorem 3, J-orthogonality is symmetric. Since the dimension is greater than two, X must be the inner-product space (Day [**2**, Theorem 6.4]).

Tapia [**6**] proved that X must be an inner-product space if the generalized inner-product is either linear or symmetric. Laugwitz [**4**] gave a geometric proof of the same result. In the following we provide another simple proof.

THEOREM 4. *For a normed linear space X the following are equivalent*:
(i) X *is an inner product space*
(ii) $\|x\| = \|y\| \Rightarrow \lim_{n\to\infty}(\|nx + y\| - \|x + ny\|) = 0$
(iii) $(x,y) = (y,x)$ *for all x and $y \in X$*
(iv) (x,y) *is linear in x for each $y \in X$.*

Proof. (i) $\Rightarrow$ (ii) is straightforward.

(ii) $\Rightarrow$ (iii) Let $\|x\| = \|y\|$. Then

$$\begin{aligned}
(x,y) &= \|x\| \ \acute{q}_+(x,y) = \|x\| \lim_{n\to\infty}(\|nx + y\| - \|nx\|) \\
&= \|y\| \lim_{n\to\infty}(\|nx + y\| - \|ny\|) \\
&= \|y\| \lim_{n\to\infty}(\|nx + y\| - \|x + ny\| + \|x + ny\| - \|ny\|) \\
&= \|y\| \lim_{n\to\infty}(\|nx + y\| + \|ny\|) \\
&= \|y\|\acute{q}_+(x,y) = (y,x).
\end{aligned}$$

If $\|x\| \neq \|y\|$, then $\| \|x\|y\| = \| \|y\|x\|$ and the argument above yields

$$\begin{aligned}(x,y) &= \|x\|\acute{q}_+(x,y) = \acute{q}_+(x,\|x\|y) - \acute{q}_+(\|y\|x,\|x\|y)\\ &= \acute{q}_+(\|x\|y,\|y\|x) + \|y\|\acute{q}_+(y,x) = (y,x).\end{aligned}$$

(iii) $\Rightarrow$ (iv). Since G-orthogonality is symmetric, by Theorem 3, X is smooth and $(x,y) = \|x\|\acute{q}_+(x,y)$ is linear in y. From this using (iii) we see that

$$a(x_1,y) + b(x_2,y) = (y,ax_1) + (y,bx_2) = (y,ax_1+bx_2) = (ax_1+bx_2,y).$$

Therefore (x,y) is linear in x for each $y \in X$.

(iv) $\Rightarrow$ (i) Let $\|x\| = \|y\| = 1$.

$$\begin{aligned}\|x+y\|\acute{q}_+(x+y,y) &= \|x+y\|\acute{q}_+(x+y,x+y-x)\\ &= \|x+y\|^2 + \|x+y\|\acute{q}_+(x+y,-x)\\ &= \|x+y\|^2 + \|x\|\acute{q}_+(x,-x) + \|y\|\acute{q}_+(y,-x)\\ &= \|x+y\|^2 + \|x\|^2 + \|y\|\acute{q}_+(y,-x)\end{aligned} \tag{1}$$

$$\begin{aligned}\|x+y\|\acute{q}_+(x+y,y) &= \|x\|\acute{q}_+(x,y) - \|y\|\acute{q}_+(y,y)\\ &= \|y\|^2 + \|x\|\acute{q}_+(x,y)\end{aligned} \tag{2}$$

From (1) and (2) we have

$$\begin{aligned}\|x+y\|^2 &= \|y\|^2 + \|x\|^2 + \|x\|\acute{q}_+(x,y) - \|y\|\acute{q}_+(y,-x)\\ &= 2 + \|x\|\acute{q}_+(x,y) - \|y\|\acute{q}_+(y,-x).\end{aligned} \tag{3}$$

Replacing y by $-y$ in (3) gives

$$\begin{aligned}\|x-y\|^2 &= 2 + \|x\|\acute{q}_+(x,-y) - \|y\|\acute{q}_+(-y,-x)\\ &= 2 + \|x\|\acute{q}_+(x,-y) + \|y\|\acute{q}_+(y,-x).\end{aligned} \tag{4}$$

Adding (3) and (4) yields

$$\|x+y\|^2 + \|x-y\|^2 = 4 + (\acute{q}_+(x,y) + \acute{q}_+(x,-y) \geq 4.$$

Thus, if in the space X (iv) holds, then

$$\|x\| = \|y\| = 1 \Rightarrow \|x+y\|^2 + \|x-y\|^2 \geq 4, \tag{S}$$

which is a characterization of inner product spaces due to Scheonberg [**5**]. That completes the proof of the theorem.

Remark. The implication (ii) of Theorem 4 is due to James [**3**, Theorem 6.3]. Our proof is different.

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