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KRIPKE MODELS FOR INTUITIONISTIC THEORIES WITH DECIDABLE ATOMIC FORMULAS

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Abstract. Some intuitionistic theories, notably Heyting's Arithmetic, have decidable atomic formulas. We show that in Kripke models of such theories, classical structures at the nodes of a Kripke model satisfy a significant fragment of corresponding theories. In particular, all consequences which are in prenex normal form hold classically.

Preliminaries

An intuitionistic theory H in a language L is a set of non-logical axioms formulated in Heyting's Predicate Calculus (HPC) with non-logical symbols from L. We shall restrict ourselves here to first order theories. $H \vdash \varphi$ means that φ is derivable from H in HPC. $H \vdash_c \varphi$ means that φ is derivable from axioms of H, in the classical Predicate Calculus. A formula $\varphi(x_1, \dots, x_n)$ is decidable in H if $H \vdash \forall x_1 \dots \forall x_n(\varphi(x_1, \dots, x_n) \vee \neg\varphi(x_1, \dots, x_n))$. For example, if Heyting's Aritmetic (HA), which has as non-logical axioms the usual Peano axioms, is formulated with function symbols $(', +, \cdot)$ the only relation symbols is equality and it is decidable. Equivalently, HA may be formulated without function symbols, with additional relation symbols: $S(x)$, $A(x, y, z)$ and $M(x, y, z)$ (for successor, addition and multiplication), and they are all decidable.

In general, Kripke model for a language L is defined as follows. Let $\langle T, \leq \rangle$ be a partially ordered set and let, with each element $s \in T$, a classical structure $\mathfrak{A}_s$ for the language L be associated, satisfying the condition: $s \leq t$ implies $\mathfrak{A}_s \subseteq^+ \mathfrak{A}_t$.

The relation $\subseteq^+$ (positive submodel) is defined by: $A_s \subseteq A_t$ (A_s is the universe of $\mathfrak{A}_s$) and for each n-ary relation symbol $R \in L$, $R^s \subseteq R^t \cap (A_s)^n$ (R^s is the interpretation of R in $\mathfrak{A}_s$) and for each n-ary function symbol $f \in L$, $f^s = f^t \restriction (A_s)^n$. We shal denote a Kripke model by $\mathfrak{M} = \langle \langle T, \leq \rangle; \mathfrak{A}_s : s \in T \rangle$. We use the symbol $\vDash$ to denote the classical satisfability relation ($\mathfrak{A}_s \vDash \varphi$). The forcing relation a $\Vdash$, is defined as usual: if $s \in T$ and $\varphi(x_1, \dots, x_n)$ is a formula of the

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language L and $a_1, \dots, a_n \in A_s$:
$s \Vdash \varphi[a_1, \dots, a_n]$ iff 1° $\varphi = R(a_1, \dots, a_n)$ is atomic and $\mathfrak{A}_s \models R(a_1, \dots, a_n)$

2°	$\varphi = \psi \wedge \chi$	and $s \Vdash \psi$ and $s \Vdash \chi$
3°	$\varphi = \psi \vee \chi$	and $s \Vdash \psi$ or $s \Vdash \chi$
4°	$\varphi = \exists x \psi(x)$	and for some $a \in A_s$, $s \Vdash \varphi[a]$
5°	$\varphi = \psi \to \chi$	and for every $t \geq s$, if $t \Vdash \psi$ then $t \Vdash \chi$
6°	$\varphi = \forall x \psi(x)$	and for every $t \geq s$ and every $a \in A_t$, $t \Vdash \psi[a]$.

The negation $\neg\varphi$ is defined as $\varphi \to \perp$, so 1° and 5° imply: $s \Vdash \neg\varphi$ iff for every $t \geq s$, $t \nVdash \varphi$.

We say that $\mathfrak{M}$ is a model of a sentence φ in L if $t \Vdash \varphi$ for every $t \in T$, or equivalently for $\langle T, \leq \rangle$ with the least element 0, if $0 \Vdash \varphi$.

A formula $\varphi(x_1, \dots, x_n)$ is decidable in $\mathfrak{M}$ if for all $t \in T$, $t \Vdash \forall x_1 \dots \forall x_n (\varphi (x_1, \dots, x_n) \vee \neg\varphi(x_1, \dots, x_n))$.

In order to simplify the appearance of the text we shall regularly suppress the valuation, but whenever we write $t \Vdash \varphi$ we assume that all free variables in φ are assigned values in A_t.

Results

LEMMA 1. *Let $\mathfrak{M} = \langle \langle T, \leq \rangle; \mathfrak{A}_t : t \in T \rangle$ be a Kripke model for the language L. $\mathfrak{M}$ is a model of a theory with decidable atomic formulas iff for any $s, t \in T$, $s \leq t$ implies $\mathfrak{A}_s$ is a submodel of $\mathfrak{A}_t$.*

Proof. Assume that all atomic formulas are decidable in $\mathfrak{M}$, i.e., if $\varphi(x_1, \dots, x_n)$ is atomic and $t \in T$, then $t \Vdash \forall x_1 \dots \forall_{x_n} (\varphi(x_1, \dots, x_n) \vee \neg\varphi(x_1, \dots, x_n))$. Let $R \in L$ be an n-ary relation symbol, $s \leq t$ and $a_1, \dots, a_n \in A_s$ and assume $\mathfrak{A}_t \models R(a_1, \dots, a_n)$, i.e. $\langle a_1, \dots, a_n \rangle \in R^t$. Then $t \Vdash R(a_1, \dots, a_n)$. Since $s \Vdash R(a_1, \dots, a_n) \vee \neg R(a_1, \dots, a_n)$ it follows that $s \Vdash R(a_1, \dots, a_n)$ and so $\mathfrak{A}_s \models R(a_1, \dots, a_n)$. Therefore $R^s = R^t \cap (A_s)^n$, so $\mathfrak{A}_s$ is a submodel of $\mathfrak{A}_t$.

Conversely, assume that $s \leq t$ implies $\mathfrak{A}_s \subseteq \mathfrak{A}_t$. Let $R \in L$ be an n-ary relation symbol and let $a_1, \dots, a_n \in A_s$. We have to show $s \Vdash R(a_1, \dots, a_n) \vee \neg R(a_1, \dots, a_n)$. If $s \nVdash R(a_1, \dots, a_n)$ by the definition of forcing $\mathfrak{A}_s \nvDash R(a_1, \dots, a_n)$. If $s \leq t$ then $\mathfrak{A}_s \subseteq \mathfrak{A}_t$ so $\mathfrak{A}_t \nvDash R(a_1, \dots, a_n)$. Therefore, for every $t \geq s$, $t \nVdash R(a_1, \dots, a_n)$, so $s \Vdash \neg R(a_1, \dots, a_n)$. $\square$

This result can easily be extended to all formulas without quantifiers.

LEMMA 2. *If in a Kripke model all atomic formulas are decidable, then in that model all quantifier-free formulas are decidable.*

Proof. Assume that in $\mathfrak{M} = \langle \langle T, \leq \rangle; \mathfrak{A}_t : t \in T \rangle$ all atomic formulas are decidable and let φ be a quantifier-free formula. The proof proceeds by induction on the logical complexity of φ. Assume the lemma holds for formulas with less than k connectives. Let $\varphi(x_1, \dots, x_n)$ have k connectives and let $t \in T$ and $a_1, \dots, a_n \in A_t$. We have three cases.

1° $\varphi = \psi \wedge x$. If $t \Vdash (\psi \wedge \chi)\ [a_1, \dots, a_n]$ we are finished. So assume $t \nVdash \psi \wedge \chi$.

This means $t \nVdash \psi$ or $t \nVdash \chi$. By induction hypothesis then $t \Vdash \neg\psi$ or $t \Vdash \neg\chi$. Therefore, for every $s \geq t$, $s \nVdash \chi$, that is $s \nVdash \psi \wedge \chi$. So $t \Vdash \neg(\psi \wedge \chi)$.

2° $\varphi = \psi \vee \chi$. Assume $t \nVdash (\psi \vee \chi)[a_1, \dots, a_n]$. This means $t \nVdash \psi$ and $t \nVdash \chi$. By induction hypothesis $t \Vdash \neg\psi$ and $t \Vdash \neg\chi$. Then, for every $s \geq t$, $s \nVdash \psi$ and $s \nVdash \chi$, so $s \nVdash \psi \vee \chi$. This means $t \Vdash \neg(\psi \vee \chi)$.

3° $\varphi = \psi \to \chi$. Assume $t \nVdash \psi \to \chi$. This means that for some $s \geq t$, $s \Vdash \psi$ and $s \nVdash \chi$. $s \Vdash \psi$ implies $t \nVdash \neg\psi$, so by induction hypothesis $t \Vdash \psi$. On the other hand $s \nVdash$ implies $t \nVdash \chi$, so again by the induction hypothesis it follows that $t \Vdash \neg\chi$. But $(\psi \wedge \neg\chi) \to \neg(\psi \to \chi)$ is a theorem of HPC, i.e. valid in all Kripke models, so $t \Vdash \neg(\psi \to \chi)$.

The case $\varphi = \neg\psi$ is a special case of 3° (when $\chi = \perp$). □

We shall prove now some results about the connection between the forcing relation at a node of some Kripke model, and the classical satisfability in the classical structure associated with that node. In [2] it was shown that a formula of the Heyting's Predicate Calculus is intuitionistically equivalent to a formula having as logical connectives only $\wedge$, $\vee$ and $\exists$ if and only if (in any Kripke model it is forced at some node iff it is classically valid in the classical model associated with that node). We can extend this class of formulas if we restrict our attention to theories with decidable atomic formulas, that is, according to the preceding lemma, to Kripke models in which the ordering relation is "submodel" (and not only "positive submodel")

Lemma 3. *Let $\mathfrak{M} = \langle\langle T, \leq\rangle; \mathfrak{A}_t : t \in T\rangle$ be a Kripke model for the language L, in which all atomic formulas are decidable and let $\varphi(x_1, \dots, x_n)$ be a quantifier-free formula in L. Then for every $t \in T$ and every $a_1, \dots, a_n \in A_t$*

$$t \Vdash \varphi[a_1, \dots, a_n] \quad \text{iff} \quad \mathfrak{A}_t \vDash \varphi[a_1, \dots, a_n]$$

Proof. The proof is by induction on the number of connectives in φ. Assume the theorem for quantifier-free formulas with less than k connectives and let φ have k connectives. The cases $\varphi = \psi \wedge \chi$ and $\varphi = \psi \vee \chi$ are a trivial consequence of the definition of forcing. We show the case $\varphi = \psi \to \chi$. Assume first $t \Vdash \psi \to \chi$. If $t \Vdash \psi$, by the definition of forcing, $t \Vdash \chi$. By induction hypothesis then $\mathfrak{A}_t \vDash \chi$, and consequently $\mathfrak{A}_t \vDash \psi \to \chi$. If $t \nVdash \psi$, by induction hypothesis on ψ, $\chi_t \nvDash \psi$. Then $\mathfrak{A}_t \vDash \neg\psi$ and so $\mathfrak{A}_t \vDash \psi \to \chi$. Conversely, assume $\mathfrak{A}_t \vDash \psi \to \chi$. This means $\mathfrak{A}_t \vDash \neg\psi$ or $\mathfrak{A}_t \vDash \chi$. If $\mathfrak{A}_t \vDash \neg\psi$, then $\mathfrak{A}_t \nvDash \psi$ and by induction hypothesis $t \nVdash \psi$. Now by Lemma 2, it follows that $t \Vdash \neg\psi$, since ψ is quantifier-free. But, $\neg\psi \to (\psi \to \chi)$ is a theorem of HPC, so $t \Vdash \psi \to \chi$. If, on the other hand, $\mathfrak{A}_i \vDash \chi$, by induction hypothesis it follows that $t \Vdash \chi$. Then again $t \Vdash \psi \to \chi$. □

We can improve this by adding existential quantifiers in front of a quantifier-free formula.

Theorem 1. *Let $\varphi(x_1, \dots, x_n)$ be an existential formula (i.e. $\varphi = \exists y_1 \dots \exists y_k\ \psi$ where ψ is quantifier-free), and let $\mathfrak{M}$ be a Kripke model in which all atomic* *formulas occurring in φ are decidable. Then for any t and any $a_1, \dots, a_n \in A_t$*

$$t \Vdash \varphi[a_1, \dots, a_n] \quad \textit{iff} \quad \mathfrak{A}_t \vDash \varphi[a_1, \dots, a_n]$$

Proof. Assume $t \Vdash \exists y_1 \dots \exists y_k \psi(y_1, \dots, y_k)$. This means that for some $b_1, \dots, b_k \in A_t$, $t \Vdash \psi[b_1, \dots, b_k]$. By Theorem 1 this holds iff $\mathfrak{A}_t \vDash \psi[b_1, \dots, b_k]$ and this implies $\mathfrak{A} \vDash \exists y_1 \dots \exists y_k \psi(y_1, \dots, y_k)$. The converse proceeds analogously. □

We can show now that one half of this equivalence holds for all formulas which are in prenex normal form. This does not imply, however, that all formulas forced at t are satisfied in $\mathfrak{A}_t$ since the Prenex Normal Form Theorem in not valid for HPC.

THEOREM 2. *Let $\varphi(x_1, \dots, x_n)$ be any formula in prenex normal form of a language. L and let $\mathfrak{M}$ be a Kripke model for L in which all atomic formulas occurring in φ are decidable. Then for any $t \in T$ and any $a_1, \dots, a_n \in A_t$*

$$t \Vdash \varphi[a_1, \dots, a_n] \quad \textit{implies} \quad \mathfrak{A}_t \vDash \varphi[a_1, \dots, a_n]$$

Proof. The proof is by induction on the number of alternations in the quantifier prefix of φ. The theorem holds for φ quantifier-free and $\sum_1$ by Lemma 3 and Theorem 1. Assume the theorem for $\sum_k$ formulas and let φ be $\prod_{k+1}$. Then $\varphi = \forall x_1 \dots \forall x_m \psi(x_1, \dots, x_m)$ where ψ is a $\sum_k$ formula. We may take $m = 1$ without loss of generality. Let $t \Vdash \forall x \psi(x)$. Then for everry $a \in A_t$, $t \Vdash \psi[a]$. Then, by induction hypothesis, $\mathfrak{A}_t \vDash \psi[a]$ for every $a \in A_t$, i.e., $\mathfrak{A}_t \vDash \forall x \psi(x)$. Now assume the theorem for $\prod_k$ formulas and let φ be $\sum_{k+1}$, that is $\varphi = \exists x \psi$ where ψ is $\prod_k$. Now $t \Vdash \exists x \psi(x)$ implies $t \Vdash \psi[a]$ for some $a \in A_t$, and by induction hypothesis, $\mathfrak{A}_t \vDash \psi[a]$, so $\mathfrak{A}_t \vDash \exists x \psi(x)$. □

Remark. That the converse implication is not generally true, even for $\prod_1$ formulas, is shown by the following simple example.

Let $\mathfrak{M} = \langle \langle \{0, 1\} \leq \rangle; \mathfrak{A}_0, \mathfrak{A}_1 \rangle$, where $\mathfrak{A}_0$ is the standard model of Peano Arithmetic (natural numbers), and $\mathfrak{A}_1$ is any model of PA $+ \varphi$, where φ is $\neg$Con (PA), i.e., $\varphi = \exists x \operatorname{Pr} f(x, \ulcorner 0 = 1 \urcorner)$. Now, $\mathfrak{A}_0 \vDash \neg\varphi$ but not $0 \vDash \neg\varphi$ (since $1 \Vdash \varphi$ and $0 \leq 1$) (cf. [3, Theorem 5.2.4.f])

COROLLARY. *Let H be an intuitionistic theory with decidable atomic formulas, with axioms in prenex normal form. Then every Kripke model of H has classical models of H at its nodes.*

THEOREM 3. *Let H be an intuitionistic theory with decidable atomic formulas. Let $\varphi(x_1, \dots, x_n)$ be any formula in the language of H and let ψ be its (classical) prenex normal form. If $H \vdash \varphi \to \psi$, then for any Kripke model $\mathfrak{M}$ of H, any node t in $\mathfrak{M}$ and any $a_1, \dots, a_n \in A_t$*

$$t \Vdash \varphi[a_1, \dots, a_n] \quad \textit{implies} \quad \mathfrak{A}_t \vDash \varphi[a_1, \dots, a_n]$$

Proof. Assume $t \Vdash \varphi$. Since $t \Vdash H$ it follows that $t \Vdash \psi$. By Theorem 2 $\mathfrak{A}_t \vDash \psi$ and since $\vdash \varphi \leftrightarrow \psi$, $\mathfrak{A}_t \vDash \varphi$. $\square$

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