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ON EDGE-COLORABILITY OF PRODUCTS OF GRAPHS

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Abstract. Let $\chi'(G)$ denote the edge-chromatic number and $\Delta(G)$ the maximum vertex degree of a graph G. A graph G is said to be *of class* 1 if $\chi'(G) = \Delta(G)$ and *of class* 2 otherwise. Some sufficient conditions for various graph products (the Cartesian, lexicographic, tensor and strong product) to be of class 1 are given.

1. Introduction and definitions. This note extends results of Himmelwright and Wiliamson [3], Kotzig [4], Mahmoodian [5] and Mohar, Pisanski and Shawe-Taylor [6, 7] concerning the edge-colorability of various graph products. Up till now we considered only the products of regular graphs, being inspired by the work of Kotzig [4]. The paper [6] summarizes our work in the regular case. However, most of the results can be extended to the products of non-regular graphs with almost no extra effort. In some cases, though, the condition of regularity is essential.

We will leave the basic definitions of graph theory to a standard reference book [2], and will limit ourselves to defining only less known terms and those wish may cause confusion.

Let v and u be vertices of a graph G. We write $v \sim u$ to denote that v and u are adjacent. Let $\chi'(G)$ denote the edge-chromatic number and $\Delta(G)$ the maximum vertex degree of a graph G. By the well-known Theorem of Vizing [9] on edge-colorability of graphs we can classify graphs into two classes. A graphs G is said to be *of class* 1 if $\chi'(G) = \Delta(G)$, and *of class* 2 if $\chi'(G) = \Delta(G) + 1$.

The *Cartesian product* $G \times H$ of graphs G and H has vertex set $V(G) \times V(H)$ and edge set

$$E(G \times H) = \{(u,v)(u',v');\ \text{either}\ (u = u'\ \text{and}\ v \sim v')\ \text{or}\ (u \sim u'\ \text{and}\ v = v')\}.$$

The *lexicographic product* $G \circ H$ of graphs G and H has vertex set $V(G) \times V(H)$ and edges $E(G \circ H) = \{(u,v)(u',v');$ either $u \sim u'$ or $(u = u'$ and $v \sim v')\}$. Note that $G \circ H$ is often written as $G[H]$.

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The *tensor product* of graphs G and H is defined as the graph $G \otimes H$ with vertex set $V(G) \times V(H)$ and the edge set

$$E(G \times H) = \{(u,v)(u',v');\ \ u \sim u' \text{ and } v \sim v'\}.$$

If G and H have the same vertex set $V = V(G) = V(H)$, and disjoint edge sets, $E(G) \cap E(H) = \emptyset$, then their *sum* $G \oplus H$ is defined as the graph having the edge set $E(G \oplus H) = E(G) \cup E(H)$. We also say that G and H are *factors* of $G \oplus H$.

The *strong product* $G * H$ is defined as $G * H = (G \circ H) \oplus (G \times H)$.

All defined products are associative thus the products of more than two graphs can be defined without confusion.

2. Results. First of all we state a simple lemma, whose trivial proof we omit.

2.1. Lemma. *Let F_1 and F_2 be two graphs which are of class* 1 *and let $F = F_1 \oplus F_2$. If*

$$\Delta(F) = \Delta(F_1) + \Delta(F_2) \tag{1}$$

then F is of class 1*.*

Now we present the generalization of Kotzig's result [4] for the cartesian product of graphs.

2.2. Theorem. *Let H be the cartesian product of graphs $G_1, G_2, \ldots, G_n$ and let one of the following two conditions be satisfied*

(i) *at least one of the graphs G_i is nontrivial and of class* 1*,*

(ii) *there exist at least two distinct indices i and j, such that G_i and G_j both contain a* 1*-factor.*

Then H is of class 1*.*

It is worth noting that the sufficient condition (i) of Theorem 2.2 was obtained before by several authors. It was first given as an exercise in Bondy and Murty's book [1, exercise 6.2.6], then obtained by Himmelwright and Williamson [3] for the case of regular graphs and letter extended by Mahmoodian [5] to the general case. At last this condition appeared in Kotzig's paper [4]. Condition (ii) can also be found there, though it is given only for the case of regular graphs. A proof similar to that in [4] can be applied to the nonregular case: Let $G_i = F_i \oplus H_i$ and $G_j = F_j \oplus H_j$, where F_i and F_j are 1-factors. Then $G_i \times G_j = (F_i \oplus H_i) \times (F_j \oplus H_j) = (F_i \times H_j) \oplus (H_i \times F_j)$. By (i), both graphs $F_i \times H_j$ and $H_i \times F_j$ are of class 1 as F_i and F_j are of class 1. It is easy to see that the factorization $(F_i \times H_j) \oplus (H_i \times F_j)$ satisfies condition (1) of Lemma 2.1. We conclude that $G_i \times G_j$ is of class 1. Now, part (i) of the theorem applies to the general case.

Let us now consider the tensor and strong products.

2.3 Theorem. *Let K be the tensor and H the strong product of graphs $G_1, G_2, \ldots, G_n$ and let at least one of the graphs G_i be nontrivial and of class* 1*. Then K and H are of class* 1*.*

We will omit the proof of Theorem 2.3, as it is the same as the one given in [7] for the regular case. Again, condition (1) of Lemma 2.1 is easily verified for all the factorizations used in the proof.

2.4. THEOREM. *The lexicographic product* $K = G_1 \circ G_2 \ldots \circ G_n$ *is of class* 1 *if at least one of the following conditions is satisfied:*

(i) G_1 *is a nontrivial graph which is of class* 1*,*

(ii) *for some* $i (2 \leq i \leq n)$ *the graph* G_i *is nontrivial and of class* 1 *and for some* $j \geq i$ *the graph* G_j *is of even order or*

(iii) *there exists two distinct indices* i *and* j*, such that the graphs* G_i *and* G_j *both contain a* 1*-factor.*

Before proving the theorem we need a lemma:

2.5. LEMMA. *Let* H *be a lexicographic product of graphs* $F_1, F_2, \ldots, F_n$ *and assume that for some* $i (1 \leq i \leq n)$ *the graph* F_i *has a* 1*-factor. Then* H *has a* 1*-factor.*

Proof. By the law of associativity of the lexicographic product

$$H = (F_1 \circ \ldots \circ F_{i-1}) \circ F_i \circ (F_{i+1} \circ \ldots \circ F_n)$$

If follows that it suffices to prove the lemma only for the case $n = 2$. But since

$$F_1 \circ F_2 = (F_1 \times F_2) \oplus (F_1 \oplus K_m), \quad m = |V(F_2)|$$

and $F_1 \times F_2$ contains a 1-factor if either F_1 or F_2 does, the lemma follows.

PROOF OF THEOREM 2.4. From

$$K = G_1 \circ (G_2 \circ \ldots \circ G_n) = (G_1 \circ \ldots \circ G_{i-1}) \circ (G_i \circ (G_{i+1} \circ \ldots \circ G_n))$$

we can conclude that it suffices to give a proof of (i) and (ii) only for the case $n = 2$. Since

$$G_i \circ \ldots \circ G_j = G_i \circ (G_{i+1} \circ \ldots \circ G_j)$$

and since by Lemma 2.5 $G_{i+1} \circ \ldots G_j$ has a 1-factor if G_j has a 1-factor, the same is true for the case (iii).

Let $K = G_1 \circ G_2$. The lexicographic product can be factored in an obvious way,

$$K = G_1 \circ G_2 = (G_1 \times G_2) \oplus (G_1 \oplus K_m), \ \ m = |V(G_2)|. \tag{2}$$

By Theorem 2.2, the cartesian product $G_1 \times G_2$ is of class 1 if at least one of the conditions (i)–(iii) is satisfied. The maximum degrees of graphs in the factorization (2) are:

$$\begin{aligned} \Delta(G_1 \circ G_2) &= m\Delta(G_1) + \Delta(G_2) \\ \Delta(G_1 \times G_2) &= \Delta(G_1) + \Delta(G_2) \\ \Delta(G_1 \oplus K_m) &= \Delta(G_1) \cdot \Delta(K_m) = \Delta(G_1). \end{aligned}$$

Therefore Lemma 2.1 can be applied and we have to prove only that the graph $G_1 \oplus K_m$ is of class 1. But this is obvious: in case (i) the graph G_i is of class 1, whereas in cases (ii) and (iii) m is even and thus K_m is of class 1 (existence of a 1-factor implies that G_2 has an even number of vertices). By Theorem 2.3 the tensor product $G_1 \oplus K_m$ is of class 1. This completes the proof.

The requirement in Theorem 2.4 (ii) that the graph $G_i \circ G_{i+1} \circ \ldots \circ G_n$ has an even number of vertices is essential. For example the graphs $C_3 \circ (K_1 \cup K_2)$ and $C_{2n+1} \circ P_{2k+1}$ $(n \leq k)$ are of class 2, though $K_1 \cup K_2$ and P_{2k+1} are of class 1. In fact there is a general construction of such graphs: Let G be any regular graph of odd order. Then for all $k's$ large enough, $G \circ P_{2k+1}$ is of class 2. To see this one observes that the given examples have only vertices of degree d and $d-1$. If the graph has an odd number of vertices, and, moreover, if the number of vertices of degree $d-1$ is less than d one can verify that such a graph must be of class 2.

3. Concluding remarks. It is interesting that Theorem 2.2 can be put in a more general setting. In particular, the results for the cartesian product can be seen as a special case of a similar result for graph bundles, obtained by Pisanski, Shawe-Taylor and Vrabec [8].

It is worth noting that the sufficient conditions of Theorems 2.2, 2.3 and 2.4 are not necessary, as it is shown in [4, 7] in the case of regular graphs.

Finally we mention some open problems concerning the classification of various graph products.

PROBLEM 1. *Let G and H both have 1-factors. Does it follow that $G \oplus H$ is of class 1?*

The answer to problem 1 is not known even for regular graphs.

PROBLEM 2. *Let H be a nontrivial graph of class 1 and of odd order and let G be of even order. Is the lexicographic product $G \circ H$ of class 1?*

PROBLEM 3. *Are the conditions* (i) *or* (ii) *of Theorem 2.2 sufficient for some more general graph products to be of class 1?*

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