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# NONEXISTENCE OF NONMOLECULAR GENERIC SETS

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**Abstract.** Generic subsets of partially ordered sets play an important role in giving significant examples of Zermelo-Fraenkel set-theoretical models. The significance of these models lies in the fact that a generic subset $G$ of a partially ordered set $P$, in general, does not exist in a model $M$ in which $P$ exists. Thus, by adjoining $G$ to $M$ an interesting extended model may ensue which has properties not shared by $M$. Thus, in considering generic extensions of set-theoretical models it is quite relevant to know whether or not a generic subset of a partially ordered set $P$ exists in the same model in which $P$ exists. In this paper, we give a necessary and sufficient condition for $P$ to have a generic subset in the same model.

Let $(P, \leq)$ be a partially ordered set. As usual, when no confusion is likely to arise, we represent $(P, \leq)$ simply by $P$. If $P$ has a *minimum* (i.e., the *smallest*) element, we represent it by 0 and we call it the *zero* element of $P$.

DEFINITION 1. *A* subset $D$ of a partially ordered set $(P, \leq)$ is called a dense (or, a coinitial) subset of $P$ if and only if for every nonzero element $x$ of $P$ there exists a nonzero element $y$ of $D$ such that $y \leq x$.

It is an interesting fact that, as shown in [1], a partially ordered set $P$ has either finitely many or else continuum many dense subsets. Clearly, every nonzero minimal element of $P$ is an element of every dense subsets. Clearly, every nonzero minimal element of $P$ is an element of every dense subset of $P$. On the other hand, if 0 of $P$ exists, 0 need not be an element of every dense subset of $P$. However, if $D$ is a dense subset of $P$ then $D - \{0\}$ as well as $D \cup \{0\}$ is a dense subset of $P$. Indeed, every superset of a dense subset of $P$ is a dense subset of $P$.

Based on the notion of a dense subset of a partially ordered set, we introduce the notion of a generic subset of a partially ordered set as follows:

DEFINITION 2. Let $(P, \leq)$ be a partially ordered set. A subset $G$ of $P$ is called a generic subset of $P$ (or, simply generic) if and only if: (1) $0 \notin G$ and $G$ is nonempty; (2) For every element $x$ and $y$ of $P$ if $x \in G$ and $x \leq y$ then $y \in G$; (3) Every two elements of $G$ have a lower bound in $G$; (4) $G$ has a nonempty intersection with every dense subset of $P$.

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We observe that by (1) and (3) every two elements of $G$ have a nonzero lower bound in $G$. Also, we observe that (1), (2) (3) imply that $G$ is a filter. Therefore, a generic subset $G$ of a partially ordered set $P$ is a filter of $P$ such that $G$ has a nonempty intersection with every dense subset of $P$.

As shown below, not every partially ordered set has a generic subset. In fact, according to Theorem 1 below, a partially ordered set has a generic subset if and only if it has a molecule. In the literature, a modified notion of a generic subset is usually considered where (4) is replaced by:

(4a) $G$ has a nonempty intersection with every dense subset of $P$ belonging to a preassigned collection $M$ of dense subsets of $P$.

If in Definition 1, condition (4) is replaced by (4a), then $G$ is called *generic over* $M$ and is denoted by $G/M$. It can be readily verified [3], that if $M$ is a denumerable collection $\{D_0, D_1, D_2, D_3, \dots\}$ of dense subsets $D_i$ of $(P, \leq)$, then $P$ always has a *generic over* $M$ subset $G$ with $p \in G$ for any nonzero element $p$ of $P$. Indeed, since $D_i$'s are dense in $P$ there exists a nonincreasing sequence:

$$\cdots \leq d_3 \leq d_2 \leq d_1 \leq d_0 \leq p$$

where $d_i \in D_i$. But then clearly $G = \{x \mid x \in P \text{ and } d_i \leq x \text{ for some } i\}$ is a generic over $M$ subset of $P$.

Definition 3. A nonzero element $m$ of a poset $P$ is called a molecule of $P$ if and only if every two nonzero elements of $P$ which are less than or equal to $m$ have a nonzero lower bound.

Clearly, every nonzero element of a simply ordered set $S$ is a molecule of $S$.

A typical partially ordered set without a molecule is provided with the following example. Let $H$ be the set of all finite sequences of the natural numbers $0, 1, 2, 3, \dots$. Let us consider the partial order $(H, \leq)$ where $p \leq q$ if and only if $p$ is an extension of $q$. Thus, $(0,3,5,2,1) \leq (0,3,5)$ and $(0,3,5,7) \leq (0,3,5)$. It can be readily verified that $(H, \leq)$ has no molecule.

Definition 4. Let $m$ be a molecule of a partially ordered set $(P, \leq)$. Then a subset $G(m)$ of $P$ is called molecular, or generated by the molecule $m$ if and only if:

$$G(m) = \{x \mid (x \in P) \wedge (\exists y)((y \neq 0) \wedge (y \in P) \wedge (y \leq m) \wedge (y \leq x))\} \tag{5}$$

Accordingly, $G(m)$ consists of those elements of $P$ each of which is greater than or equal to some element of $P$ which is less than or equal to $m$.

From (5) it obviously follows that

$$m \in G(m) \tag{6}$$

Next, we prove [cf. 4, p. 26].

Theorem 1. *A subset $G$ of a partially ordered set $(P, \leq)$ is generic if and only if $G$ is generated by a molecule of $P$.*

*Proof.* Let $G$ be a generic subset of $P$. First, we show that there exists $m \in G$ such that

(7) $\{y \mid (y \in P) \wedge (0 \neq y) \wedge (y \leq m)\} \subseteq G$ *and* $m$ *is a molecule of* $P$.

Assume on the contrary that for every $m \in G$ there exists a nonempty subset $N(m)$ of $P$ such that $z \in N(m)$ if and only if $z \leq m$ and $z \in (P - G)$. But then clearly, $P - G$ is a dense subset of $P$ which has an empty intersection with $G$, contradicting (4). Thus, the first part of (7) is established. Now, from this and (3) and the fact that $G$ is generic, it follows that $m$ is nonzero and that every two nonzero elements of $P$ which are lees than or equal to $m$ have a nonzero lower bound in $G$ (and a fortiori in $P$). Thus, $m$ is a molecule of $P$, according to Definition 3 and the proof of (7) is complete. Next, we show that $G = G(m)$. Let $x \in G$. By (3), we see that $x$ and $m$ must have a lower bound, say, $y$ in $G$. Thus, $y \leq x$ and $y \leq m$. But then, from (5) it follows that $x \in G(m)$. But then, from (5) and (7), it follows that $x \geq y$ for some $y \in G$ which by (2) implies that $x \in G$. Hence, indeed $G = G(m)$.

To complete the proof of the theorem, it remains to show that $G(m)$ as given by (5) is a generic subset of $P$.

We observe that (1) follows directly from (5). To establish (2), it is enough to observe that if $z \in G(m)$ then by (5) we see that $z \geq y$ for some nonzero $y \leq m$. Therefore, if $z \leq x$ then $x \leq y$ with $y \leq m$, which implies $x \in G(m)$. To establish (3), it is enough to observe that if $x_1 \in G(m)$ and $x_2 \in G(m)$, then by (5) we see that $y_1 \leq x_1$ and $y_2 \leq x_2$ for some nonzero $y_1 \leq m$ and nonzero $y_2 \leq m$. But since $m$ is a molecule, $y_1$ and $y_2$ have a nonzero lower bound, which by (5) is an element of $G(m)$. Hence, $x_1$ and $x_2$ have a lower bound in $G(m)$. To establish (4), let $D$ be a dense subset of $P$. But then $D$ has a nonzero element $d$ such that $d \leq m$. From (5) it follows that $d \in G(m)$ and therefore $G(m)$ has a nonempty intersection with every dense subset of $P$.

Corollary 1. *If $k$ is a nonzero minimal element of a partially ordered set $(P, \leq)$ then $k$ is a molecule of $P$ and $\{x \mid x \in P$ and $k \leq x\}$ is a generic subset of $P$.*

*Proof.* Since $k$ has no nonzero predecessors, $k$ is trivially a molecule of $P$. But then the conclusion of the corollary follows from (5).

Corollary 2. *A partially ordered set has a generic subset if and only if it has a molecule.*

This is an immediate consequence of Theorem 1 and (6). Accordingly, the partial order $(H, \leq)$ mentioned above, has no generic subset.

Let us recall the following:

Definition 5. A Boolean algebra $(A, \leq)$ is a complemented distributive lattice with a minimum 0 and a maximum 1. Moreover, a nonzero element $a$ of $A$ is called an atom of $A$ if and only if for every $x \in A$ it is the case that $x < a$ implies $x = 0$.

Furthermore, a subset $U$ of $A$ is called an ultrafilter of $A$ if and only if:

(8) $0 \notin U$; (9) For every element $x$ and $y$ of $A$ if $x \in U$ and $x \leq y$ then $y \in U$; (10) Every two elements of $U$ have a lower bound in $U$; (11) For every element $x$ of $A$ either $x \in U$ or else $x' \in U$ where $x'$ is the complement of $x$.

The reader is advised to compare (8), (9), (10), (11) with (1), (2), (3), (4). Clearly, $1 \in U$ by (11) so that $U$ is nonempty.

Lemma 1. *Let $(A, \leq)$ be a Boolean algebra. An element $a$ of $A$ is an atom of $A$ if and only if $a$ is a molecule of $A$.*

*Proof.* Let $a$ be an atom of $A$. Clearly, $a$ is a nonzero minimal element of $A$ and therefore by Corollary 1, we see that a is a molecule of $A$. Conversely, let $a$ be a molecule of $A$. To prove that $a$ is an atom of $A$ it is enough to show that $x < a$ for no nonzero element $x$ of $A$. Assume on the contrary that $x < a$ for some nonzero element $x$ of $A$. But then, since $A$ is a Boolean algebra $a - x$ exists, is nonzero and $(a - x) < a$. However, from Definition 3 it follows that $x$ and $a - x$ must have a nonzero lower bound. But this leads to a contradiction since the only lower bound of $x$ and $a - x$ in $A$ is 0. Thus, our assumption is false and $a$ is an atom of $A$.

Definitions 2 and 5 indicate that an ultrafilter of a Boolean algebra somewhat resembles a generic subset of it. However, in view of the discrepancies between (4) and (11), we must not expect that every ultrafilter of a Boolean algebra is also a generic subset of it. Indeed, in view of Theorem 1 and Lemma 1, we have:

Theorem 2. *A subset $G$ of a Boolean algebra $A$ is generic if and only if $G$ is generated by an atom of $A$ (i.e., if and only if $G$ is a principal ultrafilter of $A$).*

*Proof.* By Theorem 1 and Lemma 1, we see that $G$ must be generated by an atom $a$ of $A$ which in view of (5) implies:

$$G = G(a) = \{x \mid x \in A \ \text{ and } \ a \leq x\}$$

But then it is a routine matter to verify that the above equality implies that $G(a)$ is an ultrafilter of $A$ (as defined by (8), (9), (10), (11)) generated by the atom $a$ of $A$.

The following lemmas show some significant properties of dense subsets of a Boolean algebra.

Lemma 2. *Let $A, \leq)$ be a Boolean algebra and $D$ a dense subset of $A$. Then* $\operatorname{lub} D = 1$.

*Proof.* Clearly, 1 is an upper bound of $D$. To prove the lemma we show that every upper bound $u$ of $D$ is equal to 1. Assume on the contrary that $u$ is an upper bound of $D$ and $u < 1$. But then $1 - u$ (i.e., the complement $u'$ of $u$) is a nonzero element of $A$. However, no element of $D$ is less than or equal to $1 - u$, contradicting the denseness of $D$. Hence $u = 1$, as desired.

Lemma 3. *Let $(A, \leq)$ be a Boolean algebra and $H$ a subset of $A$ such that* $\operatorname{lub} H = 1$. *Then the subset $D$ of $A$ given by*

$$D = \{x \mid x \in A \ \text{ and } \ x \leq h \ \text{ for some } \ h \in H\} \tag{12}$$

*is a dense subset of A.*

*Proof.* We must show that for every nonzero element $p$ of $A$ there exists a nonzero element $d$ of $D$ such that $d \leq p$. Since $\operatorname{lub} H = 1$ and since $(A, \leq)$ is a Boolean algebra, we see that

$$p = p \wedge (\operatorname{lub} H) = \operatorname*{lub}_{h \in H} (p \wedge h) \tag{13}$$

Since $p \neq 0$, from (13) it follows that $(p \wedge h) \neq 0$ for some $h \in H$. Let $d = p \wedge h$. Thus, $d$ is nonzero and since $d \leq h$, we see by (12) that $d \in D$. Clearly, $d \leq p$ as desired.

As mentioned earlier, we called a nonempty subset $G$ of a partially ordered set a filter if and only if $G$ satisfies (1), (2), (3). Very often, in the literature [2], condition (3) is replaced by "*the greatest lower bound of every two elements of $G$ exists and is an element of $G$*". However, this point is immaterial for our purposes.

THEOREM. *Let $A$ be a Boolean algebra. Then a filter $G$ of $A$ is a generic subset of $G$ if and only if for every family $(a_i)_{i \in E}$ of $A$ it is the case that*

$$\operatorname*{lub}_{i \in E} a_i = 1 \ \textit{ implies } \ a_i \in G \ \textit{ for some } \ i \in E \tag{14}$$

*Proof.* Let $G$ be a generic subset of $A$. But then by Theorem 2 wee see that $A$ has an atom $a$ and $a \in G$. Now, let $\operatorname{lub}_{i \in E} a_i = 1$. Since $A$ is a Boolean algebra, we have:

$$a = a \wedge (\operatorname{lub} a_i) = \operatorname{lub}(a \wedge a_i) \ \textit{ with } \ i \in E \tag{15}$$

However, since $a$ is an atom, $a \wedge a_i = 0$ or $a \wedge a_i = a$. But then from (15) it follows that $a \wedge a_i = a$ for some $i \in E$. Hence, $a \leq a_i$ and since $a \in G$ and $G$ is a filter, $a_i \in G$ by (2). Thus, (14) is established.

Conversely, let $G$ be a filter of $A$ satisfying (14). We show that $G$ is a generic subset of $A$. To this end, in view of (4) it is enough to prove that $G$ has a nonempty intersection with every dense subset of $A$. Now, let $D$ be a dense subset of $A$. By Lemma 2 we see that $\operatorname{lub} D = 1$ and by (14) we derive that $d \in G$ for some $d \in D$. Thus, indeed $G$ has a nonempty intersection with every dense subset of $G$, as desired. In view of Theorem 2, clearly $G$ in also a principal ultrafilter of $A$.

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