PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 36 (50), 1984, pp. 51–55

THE LENGTH OF A LEMNISCATE

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Abstract. A class of two parametric real algebraic curves, is defined which contains in particular the historical lemniscata of Bernoulli. The length of such curves as a function of the parameters is studied with emphasis on its asymptotic behaviors.

1. Introduction. A. lemnicate of order n is a real algebraic curve of equation

$$|z^n - 1| = t \qquad t > 0 \tag{1}$$

where n is a positive integer, $z = x + iy$ is a complex number and $|z|$ denotes the modulo of z. The name of these curves could originate from the lemniscates are closed curves of finite length, their shape depends on whether $t < 1$, $t = 1$ or $t > 1$

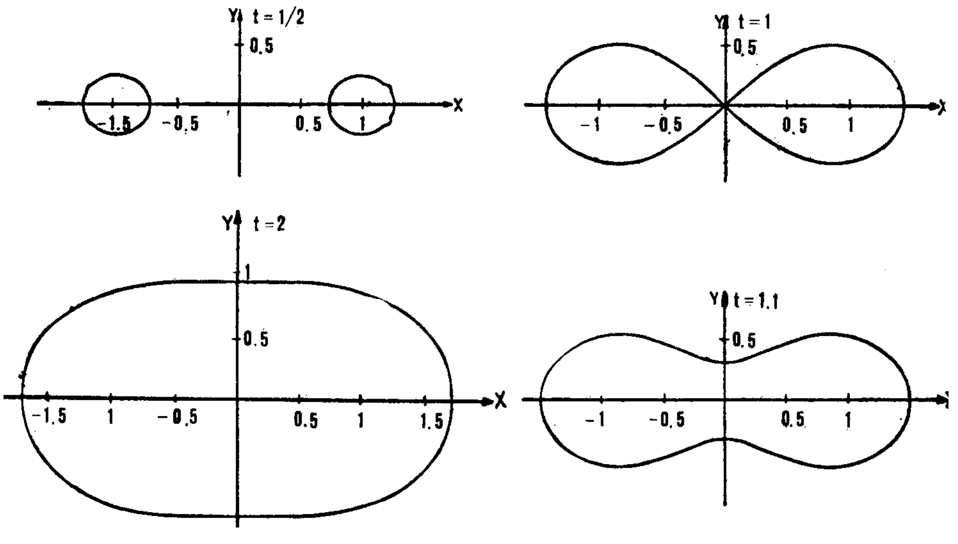

Fig. 1

AMS Subject Classification (1980): Primary 26 B 15, Secondary 51 M 25.
This paper has been presented at "XII Congresso Unione Mathematica Italiana", Perugia, 1–7 settembre 1983.

for every $n > 1$. Fig. 1 shows these three aspects in the case $n = 2$, and respectively $t = 1/2$, $t = 1$, $t = 1 \cdot 1$ and $t = 2$. The limiting case $n = 1$ need not be discussed since equation (1) represents a circumference of radius t.

Let $L_n(t)$ denote the length of a lemniscate of order n. A closed integral form of $L_n(t)$ is known, although several related problems remain still unsolved [1]. In this paper the function $L_n(t)$ is studied and its asymptotic behaviors, with respect to the parameters n and t are exploited. The expression of $L_2(t)$ i terms of elliptic integrals, which does not seem to be quoted anywhere, is also derived.

For later use it is convenient to rewrite equation (1) using polar coordinates; by setting $z = \rho e^{i\theta}$, explicitly solving for ρ, and distinguishing among values of t lower, equal or greater than 1, we have

i) $0 < t < 1$

$$\rho = \sqrt[n]{\cos n\theta \pm \sqrt{t^2 - \sin^2 n\theta}}, \quad 2\pi k - \operatorname{arc\,sin} t \le n\theta \le \operatorname{arc\,sin} t + 2k\pi, \quad 0 \le k \le n-1 \tag{3}$$

ii) $t = 1$

$$\rho = \sqrt[n]{2\cos n\theta}, \quad 4k\pi - \pi \le 2n\theta \le \pi + 4k\pi, \quad 0 \le k \le n-1 \tag{4}$$

iii) $t < 1$

$$\rho = \sqrt[n]{\cos n\theta + \sqrt{t^2 - \sin^2 n\theta}} \quad 0 \le \theta \le 2\pi \tag{5}$$

2. Explicit formulas. The length of a lemniscate is given by a definite integral which can be easily obtained from the polar representation. The derivation of the expression is made even simpler since $\rho(\theta)$ is a periodic function of θ of period $2\pi/n$. In fact the integrals can be evaluated on a period and then multiplied by n.

$$L_n(t) = n \int_{-\pi/n}^{-\pi/n} \sqrt{\rho^2 + \rho^2} d\theta \tag{6}$$

Defining

$$I_n(x) = \int_0^{\pi} \frac{\sqrt[n]{\sqrt{x + \cos^2\varphi} + \cos\varphi}}{\sqrt{x + \cos^2\varphi}} d\varphi \tag{7}$$

and using equations (3), (4) and (5), equation (6) yields

$$L_n(t) = \begin{cases} 2tI_n(t^2-1), & t > 1 \\ 2I_n(0), & t = 1 \\ 2\sqrt[n]{t} I_n(1/t^2 - 1), & 0 < t < 1 \end{cases} \tag{8}$$

From this representation it can be seen that $L_n(t)$ is a continuous function of t. In particular, we have $I_1(x) = \pi$, independent from x, so that $L_1(t) = 2t\pi$ gives the expected circumference length. Also $I_2(x)$ can be expressed by means of special elementary functions, i.e. complete elliptic integral of the first kind $K(\ )$. In order to see this we make the substitution $\cos\varphi = \sqrt{x}\sqrt{1/v^2-1}$, into equation (7) to obtain

$$I_2(x) = \frac{\sqrt[4]{4x}}{\sqrt{1+x}} \int_{\sqrt{x/(1+x)}}^{1} \frac{dv}{\sqrt{v(1-v)(v^2-x/(1+x))}} \tag{9}$$

and applying the classical bilinear transformations of elliptic integrals [2, pp. 82–86] we get

$$I_2(x) = \frac{4\sqrt{2}}{\sqrt[4]{1+x}} \frac{1}{\sqrt{2}+\sqrt{1+\sqrt{x/(1+x)}}} K\left(\frac{1-\sqrt{x/(1+x)}}{\left[\sqrt{2}+\sqrt{1+\sqrt{x/(1+x)}}\right]^2}\right). \tag{10}$$

Hence the length of the lemniscate of order two results in

$$L_2(t) = \begin{cases} \frac{8\sqrt{2t}}{\sqrt{2}+\sqrt{1+\sqrt{1-1/t^2}}} K\left(\frac{1-\sqrt{1-1/t^2}}{[\sqrt{2}+\sqrt{1+\sqrt{1-1/t^2}}]^2}\right) & t>1 \\ 8\sqrt{2}/(1+\sqrt{2})K(1/(3+2\sqrt{2})) & t=1 \\ \frac{8\sqrt{2t}}{\sqrt{2}+\sqrt{1+\sqrt{1-t^2}}} K\left(\frac{1-\sqrt{1-t^2}}{[\sqrt{2}+\sqrt{1+\sqrt{1-t^2}}]^2}\right) & 0 \le t < 1. \end{cases} \tag{11}$$

In general, it seems that the integral $I_n(x)$ and in turn $L_n(t)$ cannot be expressed in terms of either elementary or special functions. Hereafter $L_n(t)$ will be studied with reference to its integral representation. Anyway, the particular integral $I_n(0)$ is computable in terms of gamma functions (see [3]).

$$I_n(0) = \int_0^x \frac{\sqrt{|\cos\varphi| + \cos\varphi}}{|\cos\varphi|} d\varphi = \int_0^\pi (\cos\varphi)^{-1+1/n} d\varphi = \frac{\pi\Gamma(1/n)}{\Gamma^2(1/2+1/2n)}. \tag{12}$$

Furthermore, by means of the relation $\Gamma(z)\Gamma(1-z) = \pi\sin(z\pi)$, another expression, useful for exploiting the asymptotic behavior of $L_n(1)$ as $n \to \infty$, is obtained

$$L_n(1) = \frac{2\pi^2}{\sin(\pi/n)\Gamma(1-1/n)\Gamma^2(1/2+1/2n)}. \tag{13}$$

3. Asymptotic results. In this section the asymptotic expressions concerning the function $L_n(t)$ will be obtained, both for fixed t letting $n \to \infty$, and for fixed n letting either $t \to \infty$ or $t \to 0$. Let us first consider $t < 1$. In this case $L_n(t)$ explicitly results in the following

$$L_n(t) = 2t \int_0^\pi \frac{\sqrt[n]{\sqrt{1-t^2\sin^2\varphi} + t\cos\varphi}}{\sqrt{1-t^2\sin^2\varphi}} d\varphi \tag{14}$$

Therefore, fixing t, as $n \to \infty$ the integral uniformly tends to $2K(t^2)$, so that

$$L_n(t) \to 4tK(t^2) \tag{15}$$

while as $t \to 0$, the integral can be evaluated by means of a series expansion to give

$$L_n(t) \to 2\pi t + \pi^2(1 - 1/n)^2 t^3/4 \tag{16}$$

The case $t = 1$ and $n \to \infty$ is simply disposed of: using equation (13) we obtain

$$L_n(1) \to 2n \tag{17}$$

Now let us consider $t > 1$, here again $L_n(t)$ explicitly results in the following

$$L_n(t) = 2\sqrt[n]{t} \int_0^\pi \frac{\sqrt[n]{\sqrt{1 - (\sin^2\varphi)/t^2} + \cos\varphi/t}}{\sqrt{1 - (\sin^2\varphi)/t^2}} d\varphi \tag{18}$$

Therefore for t fixed, as $n \to \infty$ the integral uniformly tends to $2K(1/t^2)$, so that

$$L_n(t) \to 4K(1/t^2) \tag{19}$$

while as $t \to \infty$ the integral tends to π so that

$$L_n(t) \to 2\pi \sqrt[n]{t} \tag{20}$$

The equations (15), (17) and (19) show the singular behavior of the function $L_n(t)$ in the point $t = 1$. Such a function for $t = 1$ diverges with n, while for $t \neq 1$ the function values, at the limit as $n \to \infty$, are constant and independent from n. In order to see what kind of singularity the point $t = 1$ is, and for exhibiting the qualitative shape of $L_n(t)$, it is useful to compute the derivative

$$dL_n(t)/dt = \begin{cases} 2I_n(t^2 - 1 + 4t^2 I_n(t^2 - 1) & t > 1 \\ 2t^{-1+1/n} I_n(1/t^2 - 1)_/n - 4t^{-3+1/n} I_n(1/t^2 - 1) & 0 < t < 1 \end{cases} \tag{21}$$

where $I_n(x)$ means $dI_n(x)/dx$.

It can be seen that for $0 < t < 1 dL_n(t)/dt > 0$, while on the other hand, for $t > 1$, the derivative presents at least one change of sign. Moreover we have

$$\lim_{t \to -1} dL(t)/dt = +\infty \qquad \lim_{t \to +1} dL_n(t)/dt = -\infty$$

showing that in $t = 1$ the curve representing $L_n(t)$ has a cusp.

4. Conclusions. The qualitative shape of $L_n(t)$, $n \geq 2$, is shown in Fig. 2 whose drawing relies on the following facts

1. $L_n(0) = 0$
2. $L_n(t)$ tends to infinity with t;
3. $L_n(t)$ has at least one minimum in the range $t > 1$;

4. $L_n(t)$ is monotone increasing in the range $0 < t < 1$;

5. $(1, L_n(1))$ is a cuspidal point with vertical tangent.

The function $L_n(t)$ has been almost completely characterized, but certain problems are still unsolved. It seems likely that $L_n(t)$ presents only one minimum, but this fact is to be proved; moreover, if this conjecture is true, a second problem can be raised concerning the asymptotic behavior of this minimum for n tending to infinity.

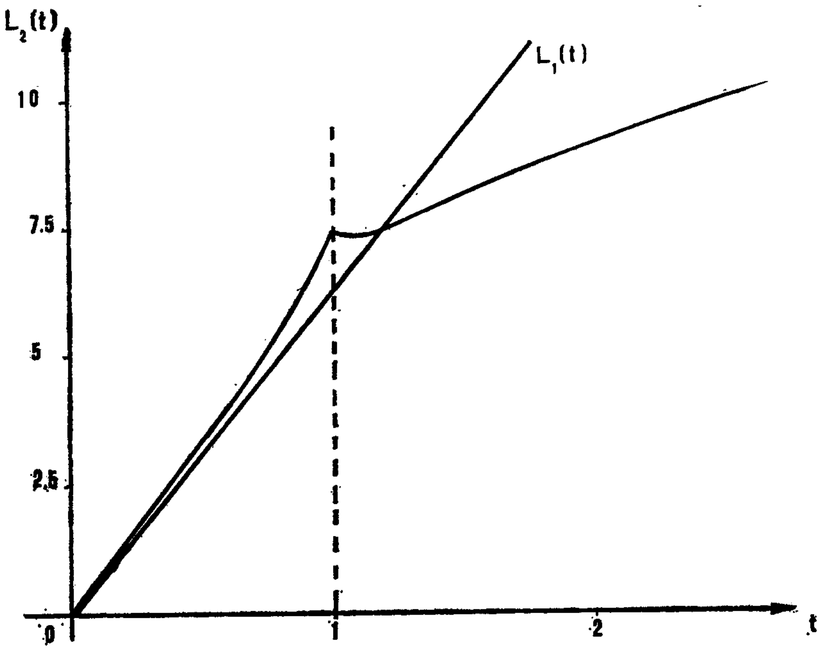

Fig. 2

Finally, a third open problem is to show whether or not $I_n(x)$, for n greater than 2 is expressible in terms of classical special functions.

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(Received 24 10 1983)