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ON CONVERGENCE DOMAINS OF FUNCTION METHODS

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Abstract. We consider function convergence methods and the possibility of giving analogous formulations of Schur's Theorem for matrix convergence methods. So we give a class of function methods and conditions under which an analogue of Schur's Theorem is valid.

Let $F = (f_n(x))$ be a sequence of functions which map a set X into the complex plane C and let $\mathcal{F}$ be a filter on X.

DEFINITION 1. The complex sequence (s_n) converges to s by the function convergence method $(X, F, \mathcal{F})$ (briefly, $(s_n)F$-converges to s) if the series[1] $\sum_n f_n(x)s_n$ converges for every $x \in X$ and

$$s = \lim_{x,\mathcal{F}} \sum_n f_n(x)s_n.$$

If X is the set N of natural numbers and $\mathcal{F}$ is a Frechet filter ($G \in \mathcal{F}$ iff G is the complement of a finite subset in N), then we have the matrix convergence method associated with matrix $A = (a_{kn})$ where $a_{kn} = f_n(k)$ $(k, n \in N)$.

DEFINITION 2. The complex sequence (s_n) converges to s by matrix convergence method (briefly, (s_n) A-converges to s) if the series $\sum_n a_{kn}s_n$ converges for every $k \in N$ and

$$s = \lim_{k\to\infty} \sum_n a_{kn}s_n.$$

For the matrix convergence method defined by a matrix A Schur has proved the following result (see [3]).

THEOREM A. *Every bounded sequence is A-convergent iff the following conditions are satisfied:*

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[1] $\sum_n A_n$ stands for $\sum_{n=1}^{\infty} A_n$ unless stated otherwise.

1° $\lim_{n\to\infty} a_{nk} = a_k$ $(k \in N)$

2° $\sum_k |a_{nk}| < \infty$ $(n \in N)$

3° $\lim_{n\to\infty} \sum_k |a_{nk} - a_k| = 0.$

It is easy to see that the conjunction of conditions 1°, 2° and 3° is equivalent to the conjunction of 1° and

4° $\sum |a_{nk}|$ converges uniformly by n.

So the following result is valid (see (1]).

Theorem B. *Every bounded sequence is A-convergent iff conditions* 1° *and* 4° *are satisfied.*

Let us try to formulate analogously the preceding theorems for function convergence methods. For the sake of simplicity we will consider the case when $X = (a, b](-\infty \le a < b < +\infty)$ and $\mathcal{F}$ consists the intersections of $(a, b]$ and the neighborhoods of a in the following result.

Theorem A'. *The convergence domain $((a, b], F)^c$ of the function convergence method $((a, b], F)$ includes the set of all bounded sequences T_b iff the following conditions are satisfied:*

1′ $\lim_{x\to a+0} f_k(x) = g_k$ $(k \in N)$

2′ $\sum_k |f_k(x)| < \infty$ $(x \in X)$

3′ $\lim_{x\to a+0} \sum_k |f_k(x) - g_k| = 0.$

There is some difficulty to formulate analogously Theorem B for function convergence methods. Namely, one can observe many possibilities for adapting condition 4° such as

4′ the series

$$\sum_k |f_k(x)| \tag{$*$}$$

converges uniformly on $(a, b]$;

4′′ the series $(*)$ converges on $(a, b]$ and converges uniformly on $(a, c]$ where c is fixed and $c \in (a, b]$;

4′′′ the series $(*)$ converges on $(a, b]$ and converges uniformly on $\cup_p\{x_p\}$ for every sequence (x_p) on $(a, b]$ converging to a.

If the function convergence method $((a, b], F)$ is continuous in the sense of [4], then the following holds (see [2]).

Theorem B.′ *The convergence domain of continuous function convergence method contains the set of all bounded sequences iff conditions* 1′ *and* 4′ *are satisfied.*

If a functional convergence method is not continuous, then Theorem B' is not true. Moreover we have following fact.

Proposition. *There exists a function convergence method whose convergence domain contains the set of all bounded sequences but the series $(*)$ does not converge uniformly on any $(a, c]$.*

PROOF: Set

$$x_n = 2^{-n}, y_{nk} = x_{n+1} + x_{n+k+1} \qquad (n, k \in N).$$

Let us define a sequence of continuous functions of $(0,1]$:

$$u_k(x) = d(x, (0,1] - A_k) \qquad (k \in N)$$

where $A_k = \cup_{n=0}^{\infty}(y_{nk+1}, y_{nk})$ and d is usual distance on the real line.

The sequence $(u_k(x))$ has the following properties:

(a) $0 \le u_k(x) \le x$ for $x \in (0,1]$ and $k \in N$;

(b) for every $x \in (0,1]$ there exists a $K(x) \in N$ such that $k \ne K(x) \Rightarrow u_k(x) = 0$ for every $k \in N$;

(c) $u_k(x)$ converges pointwise to 0 on $(0,1]$;

(d) $u_k(x)$ converges uniformly to 0 on $\cup_p\{x_p\}$, where (x_p) is an arbitrary sequence on $(0,1]$ converging to 0;

(e) for any a where $0 < a \le 1$, $u_k(x)$ does not converge uniformly on $(0,a]$.

Property (a) follows immediately from the definition of $u_k(x)$.

If $x \in (0,1]$, then there exist $s,t \in N$ such that $x \in [y_{st+1}, y_{st}]$. In this case $k \ne t$ implies $u_k(x) = 0$. Hence, $K(x) = t$ and property (b) holds.

Now property (c) follows immediately from (b).

Let $x_p \to 0$ and $\varepsilon > 0$. The inequality $x_p \ge \varepsilon$ is true for at most a finite number of values of p, for instance $p_1, p_2, \dots, p_r$. Let us take

$$k(\varepsilon) = \max\{K(x_{p_1}), K(x_{p_2}), \dots, K(x_{p_r})\}$$

where $K(x_{p_i})(1 \le i \le r)$ are determined by property (b). Then $u_n(x_{p_i}) = 0$ for every $n > k(\varepsilon)$ and $1 \le i \le r$. Since we have $0 < x_p < \varepsilon$ whenever $p \ne p_i(1 \le i \le r)$, we have that for every $n \not\in N$ and $p \ne p_i(1 \le i \le r)$ the following holds:

$$\theta \le u_n(x_p) \le x_p < \varepsilon.$$

Finally, we conclude that for every $\varepsilon > 0$ there exists a $k(\varepsilon)$ such that $n > k(\varepsilon)$ implies $u(x_p) < \varepsilon$ for every $p \in N$. Thus (d) is proved.

To prove (e) it suffices to show that for every $0 < a \le 1$ there exists a $\varepsilon(a) > 0$ and a sequence (z_k) from $(0,a]$ such that

$$\lim_{k\to\infty} u_k(z_k) = \varepsilon(a).$$

Indeed, for a fixed number a we choose m such that $a > 2^{-m-1}$ and set $z_k = (y_{mk+1} + y_{mk})/2$. As $z_k \to 2^{-m-1}$, all but a finite number of members of the sequence (z_k) belong to $(0,a]$. Moreover we have

$$\lim_{k\to\infty} u_k(z_k) = \lim_{k\to\infty} z_k = 2^{-m-1}.$$

Now, property (e) is proved by taking $\varepsilon(a) = 2^{-m-1}$.

The properties of the function sequence $(u_n(x))$ enable us to construct a function convergence method such that:

(i) the convergence domain contains all the bounded sequences;

(ii) the method does not satisfy condition $4''$.

Set $a = 0$, $b = 1$, $f_1(x) = u_1(x)$, $f_{k+1}(x) = u_{k+1}(x) - u_k(x)$ $(k \in N)$. The properties mentioned before imply:

(a$'$) $\lim_{x\to+0} f_k(x) = 0 \quad (k \in N)$

(b$'$) $\sum_k |f_k(x)| \le 2x \quad (x \in (0,1])$;

(c$'$) $\lim_{x\to+0} \sum_k |f_k(x) - 0| \le 2l \lim_{x\to+0} x = 0$.

Hence, the convergence domain of our method contains the set of all bounded sequences by virtue of Theorem A$'$.

Since

$$\sum_{k=0}^{n} f_k(x) = u_n(x) \quad (n \in N)$$

we conclude that the series $\sum_k f_k(x)$ does not converge uniformly on any interval $(0,a]$, $0 < a \le 1$.

Consequently, the series $\sum_k |f_k(x)|$ has the same property and this completes the proof of our proposition.

Remark 1. The previous example shows that there exists a function convergence method whose functions are continuous but the method is not continuous in the sense of [2].

Further, we can easily prove the following criterion:

Let the convergence domain of $((a,b],F)$ contain the set of all bounded sequences. If for every $\varepsilon > 0$ there exists a sequence (y_n) such that

$$0 < y_n - a < \varepsilon \ \ (n \in N) \ \text{ and } \ \lim_{n\to\infty} \inf f_n(y_n) = d > 0$$

then this method is not continuous.

When $f_n(x) = a_n x^n g(x)$ it is easy to check whether this method is continuous or not. If $f_n(x)$ are arbitrary functions, the checking is quite difficult. Let us consider the following class of functional convergence methods.

Definition 3. The function convergence method $((a,b],F)$ is an L-type method if the series $(*)$ converges on $(a,b]$ and if this series converges uniformly on any interval $[a_1,c]$, where $a < a_1 \le c$ and c is fixed number in $(a,b]$.

Remark 2. If there exists a decreasing sequence (a_n) from $(a,b]$ converging to a such that

$$\sum_{j=k}^{\infty} |f_j(x)| \qquad (k \in N) \tag{$(**)$}$$

is monotone on (a_{m+1}, a_m) $(m \in N)$, then this method is an L-type method.

According to the previous remark we can conclude that the following methods are L-type methods.

Example 1. Let $A = (a_{ij})$ be an infinite matrix with the property $\sum_j |a_{ij}| < \infty$ $(i \in N)$. Let us define the sequence of functions $F = (f_j(x))$:

$$f_j(x) = \begin{cases} a_{ij} & \text{for } x = i \\ b_{ij}a_{ij} & \text{for } x \in (i, i+1) \end{cases} \qquad (i \in N)$$

where $|b_{ij}| < B_1$.

Condition $(**)$ is satisfied because $f_j(x)$ are constant for $x \in (i, i+1)$.

If $f_j(x)$ are continuous on $(i, i+1]$, then this method is equivalent to the matrix convergence method associated with the matrix A.

Example 2. Let us consider the Abel method. In this case $f_j(x) = (1 - x)x^{j-1}$ $(j \in N)$ on the interval $[0, 1)$. The corresponding sum in the condition $(**)$ is equal x^{k-1}. This function is obviously monotone on $[0, 1)$.

Theorem B$''$. *The convergence domain of an L-type method $((a, b], F)$ contains all the bounded sequences iff conditions $1'$ and $4''$ are satisfied.*

In the proof we need the following simple lemma which might be interesting by itself.

Lemma. *Let $u_k(x)$ be a function sequence converging pointwise to $u(x)$ on $(a, b]$. Then the conditions*

(1) *$u_k(x)$ converges uniformly to $u(x)$ on $\cup_k\{a_k\}$, where $a_k \in (a, b]$ and $a_k \to a$*

(2) *$u_k(x)$ converges uniformly to $u(x)$ on each interval $[c, b]$ where $a < c \leq b$ imply the uniform convergence of the sequence $(u_k(x))$ to $u(x)$ on $(a, b]$.*

Proof. Let us suppose that $(u_k(x))$ does not converge uniformly on $(a, b]$. Then there exists a convergent sequence (z_k) from $(a, b]$ such that

$$\lim_{k\to\infty} \inf |u_k(z_k) - u(z_k)| > 0.$$

There are two possibilities: either (z_k) converges to a, or (z_k) does not converge to a. It is easy to see that the first possibility contradicts condition (1) and the second one contradicts condition (2).

Proof of the Theorem: Let us suppose that conditions $1'$ and $4''$ are satisfied. If $(s_j) \in T_b$, then $\sum_j f_j(x)s_j$ converges for every $x \in (a, b]$. Moreover this series converges uniformly on $(a, c]$ and we have

$$\lim_{x\to a+o} \sum_j f_j(x)s_j = \sum_j g_j s_j$$

for every $(s_j) \in T_b$. So, conditions $1'$ and $4''$ are sufficient for the convergence domain of an L-type method to contain all the bounded sequences. (Obviously, $1'$ and $4''$ are sufficient for $((a, b], F)^c \supseteq T_b$ in general.)

Let us prove that conditions $1'$ and $4''$ are necessary. Therefore, suppose that every bounded sequence (s_k) is convergent by the L-type method $((a,b],F)$. From the results concerning function convergence methods in general we can see that condition $1'$ and the first part of condition $4''$ are satisfied.

Suppose that (x_i) is a sequence from $(a,b]$ converging to a. Now, we can define a matrix convergence method by

$$a_{ij} = f_j(x_i) \qquad (i,j \in N) \tag{$(***)$}$$

The convergence domain of this method contains T_b by our assumption. Applying Theorem B, we have that the series $\sum_j |f_j(x_i)|$ converges uniformly by $i \in N$. Since (x_i) is an arbitrary sequence, condition (1) of the Lemma is satisfied. Condition (2) of the Lemma is satisfied by definition of L-type method. Hence, we conclude that the second part of condition $4''$ is satisfied and this completes the proof.

Finally, for general function convergence methods we have the following theorem.

THEOREM B$'''$. *The convergence domain of a function convergence method contains all the bounded sequences iff conditions* $1'$ *and* $4''$ *are satisfied.*

PROOF OF THE SUFFICIENCY PART: If $T_b \subseteq ((a,b],F)^c$, then condition $1'$ is satisfied in virtue of Theorem A'. Let (x_p) be a sequence on $(a,b]$ converging to a. Let us define the matrix convergence method by the matrix $A=(a_{ij})$ described in $(***)$. Obviously, $A^c \supseteq ((a,b],F)^c \supseteq T_b$ and we get condition $4'''$ by applying Theorem B.

PROOF OF THE NECESSITY PART: Let (x_p) and A be the same as in the preceding part. If (s_n) is a bounded sequence, then the series $\sum_n f_n(x_p)s_n$ converges uniformly in p, because the series $\sum_n |f_n(x_p)|$ converges uniformly by p (Theorem B). Hence we have:

$$\lim_{p\to\infty} \sum_n f_n(x_p)s_n = \sum_n g_n s_n.$$

Finally, we see that the following limit exists:

$$\lim_{x\to a+0} \sum_n f_n(x)s_n.$$

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