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ON THE CURVATURE COLLINEATION IN FINSLER SPACE

S.P. Singh

Abstract. We define $\bar{x}^i = x^i + v^i(x)\delta t$ as the h-curvature collineation of a Finsler space, supposing that the Lie derivative of Bervald's curvature tensor is equal to zero. Then we prove that every motion and every homothetic transformation admitted in a Finsler space are H-curvature collineations. Some special cases are also discussed.

The motions in a Finsler space have been studied by many authors. Hiramatu [4] has obtained a group of homothetic transformation in a Finsler space with the help of Lie-derivatives. The purpose of the present paper is to study the curvature collineation in a Finsler space.

1. Introduction

We consider an n-dimensional Finsler space F_n equipped with the fundamental function $F(x^i, \dot{x}^i)$. The covariant derivative of a tensor field, say X^i in the sense of Berward [3], is given by

$$X^i_{(k)} = \delta_k x^i - \dot{\delta}_h x^i \dot{\delta}_k G^h + x^h G^i_{hk}, \tag{1.1}$$

where $G^h(x, \dot{x})$ is a positively homogeneous function of degree two in $\dot{x}^i$ and the connection coefficient is given by

$$G^i_{jk} = \dot{\delta}^2_{jk} G^i \tag{1.2}$$

The corresponding curvature tensor field H^i_{jkh} of F_n is defined as

$$H^i_{hjk} = \delta_h G^i_{jk} - \delta_j G^i_{hk} + G^r_{hj} G^i_{rk} - G^r_{hk} G^i_{rj} + G^i_{rhk} \dot{\delta}_j G^r - G^i_{rhj} \dot{\delta}_k G^r. \tag{1.3}$$

The commutation formula involving the curvature tensor field H^i_{jkh} are as follows:

$$2T_{[(h)(kj)]} = T_{(h)(k)} - T_{(k)(h)} = -\dot{\delta}_i T H^i_{hk} \tag{1.4}$$

$$2T^i_{j[(h)(k)]} = -\dot{\delta}_r T^i_j H^r_{hk} - T^i_r H^r_{jhk} + T^r_j H^i_{rhk}, \tag{1.5}$$

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In a Finsler space F_n, we consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\delta t, \tag{1.6}$$

where $v^i(x)$ is a contravariant vector field. Then we have a deformed space with connection $G^i_{jk} + (\mathcal{L}_v G^i_{jk})\delta t$, where $\mathcal{L}_v$ denotes the Lie-derivative with respect to $v^i(x)$.

With respect to Lie-derivative for any tensor T^i_{jk}, we have

(a)

$$\begin{aligned}(\mathcal{L}_v T^i_{jk(l)}) - (\mathcal{L}T^i_{jk})_{(l)} &= (\mathcal{L}_v G^i_{rl})T^r_{jk} - (\mathcal{L}_v G^r_{jl})T^i_{rk} \\ &\quad - (\mathcal{L}_v G^r_{kl})T^i_{jr} - (\mathcal{L}_v G^r_{lp})\dot{x}^p \dot{\delta}^r T^{ik}_j\end{aligned} \tag{1.7}$$

(b)

$$\dot{\delta}_t(\mathcal{L}_v T^i_{jk}) - \mathcal{L}_v(\dot{\delta}_t T^i_{jk}) = 0. \tag{1.7}$$

The Lie-derivative of the curvature tensor H^i_{jkh} is given by

$$(\mathcal{L}_v G^i_{jh})_{(k)} - (\mathcal{L}_v G^i_{kh})_{(j)} = \mathcal{L}_v H^i_{hjk} + (\mathcal{L}_v G^r_{kl})\dot{x}^l G^i_{rjh} - (\mathcal{L}_v G^r_{jl})\dot{x}^t G^i_{rkh}. \tag{1.8}$$

2. The curvature collineation in a Finsler space

Definition. In a Finsler space F_n, if the curvature tensor field H^i_{jkh} satisfies the relation

$$\mathcal{L}_v H^i_{jkh} = 0, \tag{2.1}$$

with respect to the vector field $v^i(x)$, the infinitesimal transformation (1.6) is called an *H-Curvature Collineation.*

The infinitesimal transformation (1.6) is called an affine motion if it satisfies the relation $\mathcal{L}_v g_{ij} = 0$. If (1.6) is to be an affine motion, it is necessary and sufficient that we have

$$\mathcal{L}_v G^i_{jk} \equiv v^i_{(j)(k)} + H^i_{jkh}v^h + G^i_{jkh}v^h_{(r)}\dot{x}^r = 0. \tag{2.2}$$

Applying (2.2) in (1.8), we get $\mathcal{L}_v H^i_{jkh} = 0$. Hence we have

Theorem 2.1. *Every motion admitted in Finsler space F_n is an H-curvature Collineation.*

In view of identity (1.7) (b) for H^i_{jkh} and (2.1), we obtain $\mathcal{L}_v \dot{\delta}_t H^i_{jkh} = 0$. Accordingly we have

Lemma 2.1. *In a Finsler space F_n which admits the H-Curvature Collineation, the partial derivative of the curvature tensor H^i_{jkh} is Lie-invariant.*

By virtue of the identity (1.5) for the tensor field H^i_{jkh}, we find

$$\begin{aligned}H^i_{jkh(l)(m)} - H^i_{jkh(m)(l)} &= -\dot{\delta}_r H^i_{jkh}H^r_{lm} + H^r_{jkh}H^i_{rlm} \\ &\quad - H^i_{rkh}H^r_{jlm} - H^i_{jrh}H^r_{klm} - H^i_{jkr}H^r_{hlm}.\end{aligned} \tag{2.3}$$

Now applying $\mathcal{L}_v$ operator on both sides of (2.3) and using (2.1) and Lemma 2.1, we obtain

$$\mathcal{L}_v H^i_{jkh(l)(m)} = \mathcal{L}_v H^i_{jkh(m)(l)}, \quad \text{if} \quad \mathcal{L}_v \dot{x}^p - 0. \tag{2.4}$$

Hence consequently, we state

THEOREM 2.2. *In a Finsler space F_n which admits an H-curvature collineation, the relation (2.4) holds if $\mathcal{L}_v \dot{x}^p = 0$.*

Using the identity (1.7) (a) for the curvature tensor H^i_{jkh} and noting (2.1), we get

$$\begin{aligned}\mathcal{L}_v(H^i_{jkh(l)}) = (\mathcal{L}_v G^i_{rl})H^r_{jkh} - (\mathcal{L}_v G^r_{jl})H^i_{rkh} - (\mathcal{L}_v G^r_{kl})H^i_{jrh} \\ - (\mathcal{L}_v(G^r_{hl})H^i_{jkr} - (\mathcal{L}_v G^r_{lp})\dot{x}^p \dot{\delta}_r H^r_{jkh}.\end{aligned} \tag{2.5}$$

If the H-curvature collineation (1.6) is a motion $\mathcal{L}_v G^i_{jk} = 0$ is satisfied. In the case (2.5) reduces to $\mathcal{L}_v(H^i_{jkh(l)} = 0$.

Accordingly we have

THEOREM 2.3. *When an H-curvature collineation admitted in F_n becomes a motion in the same space, the Lie-derivative of the tensor $H^i_{jkh(l)}$ vanishes identically.*

A necessary and sufficient condition that the infinitesimal transformation (1.6) be a homothetic transformation [4] is that the relation $\mathcal{L}_v g_{ij} = 2Cg_{ij}$, where C is a constant, holds. In case F_n admits a homothetic transformation (1.6), the condition $\mathcal{L}_v G^i_{jk} = 0$ holds. Hence immediately from (1.8) we obtain $\mathcal{L}_v H^i_{jkh} = 0$. We state

THEOREM 2.4. *Every homothetic transformation admitted in a Finsler space F_n is an H-curvature collineation.*

3. Special cases

We consider the following cases which are of interest:

(a) *Contra Field.* In a Finsler space F_n, if the vector field $v^i(x)$ satisfies the relation

$$v^i_{(j)} = 0, \tag{3.1}$$

the vector field $v^i(x)$ determines a contra field. Here we consider a special H-curvature collineation:

$$\bar{x}^i = x^i + v^i(x)\delta t, \ \text{ with } \ v^i_{(j)} = 0. \tag{3.2}$$

In this case, if (3.2) is a motion, the equation (2.2) yields

$$\mathcal{L}_v G^i_{jk} \equiv H^i_{jkl} v^l = 0. \tag{3.3}$$

Also the integrability condition of (2.2) becomes

$$\mathcal{L}^v H^i_{jkh} \equiv H^i_{jkh(l)} v^l = 0. \tag{3.4}$$

Accordingly we state

THEOREM 3.1. *In a Finsler space F_n which admits an H-curvature collineation, if the vector field $v^i(x)$ spans a contra field, the conditions $H^i_{jkl}v^l = 0$ and $H^l_{jkh(l)}v^l = 0$ necessarily hold.*

A non-flat Finsler space F_n in which there exists a non-zero vector field whose components K_m are positively homogeneous function of degree zero in $\dot{x}^i$, such that the curvature tensor field H^i_{jkh} satisfies

$$H^i_{jkh(l)} = K_l H^i_{jkh}, \tag{3.5}$$

is called a recurrent Finsler space ([5], [7]).

Applying (3.5) in (3.4), we find

$$H^i_{jkh} K_l v^l = 0. \tag{3.6}$$

Since F_n is anon-flat space, we get

$$K_l v^l = 0, \tag{3.7}$$

which is a necessary condition. From Theorem 2 [6, p. 264] it is a sufficient condition also. Thus we state

THEOREM 3.2. *In a recurrent Finsler space F_n which admits an H-curvature collineation for the vector field $v^i(x)$ to span a contra-field it is necessary and sufficient that $H^i_{jkl}v^l = 0$ and $K_l v^l = 0$.*

(b) *Concurrent Field.* In a Finsler space F_n, if the vector field $v^i(x)$ satisfies the relation

$$v^i_{(j)} = K\delta^i_j, \tag{3.8}$$

where K is a non-zero constant, the vector field $v^i(x)$ is said to determine a concurrent field.

We consider the H-curvature collineation of the form

$$\bar{x}^i = x^i + v^i(x)\delta t, \quad v^i_{(j)} = K\delta^i_j. \tag{3.9}$$

Applying the latter of (3.9) in (2.2), we obtain $H^i_{jkl}v^l = 0$.

The covariant differentiation of it, in view of (3.5) and (3.8), gives $KH^i_{jkh} = 0$. But K is a non-zero constant, hence it yields $H^i_{ikh} = 0$. This contradicts our assumption that the Finsler space F_n is non-flat. Accordingly we state

THEOREM 3.3. *A general recurrent Finsler space F_n does not permit an H-curvature collineation of the form (3.9).*

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Department of Mathematics
Kenyatta Univ. College,
Box 43844, Nairobi,
Kenya

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