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NOTE ON THE NUMBER OF SEQUENCES WITH GIVEN COMPLEXITY

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Abstract. Kolmogorov in 1964 has defined the notion of complexity of a finite binary sequence. In this paper some properties of the number of sequences with given complexity are considered.

Let S be the set of all finite binary sequences, and let $l(x)$, $x \in S$ be the length of x. Let $K(x)$ be the complexity of x with respect to some optimal function F_0^1 and let $K(x/y)$ be the conditional complexity of x given y (see [1] – [3]). It is well known that the complexity of most of the sequences is close to their length (see [2]), or

$$(\exists c)(\forall x)(K(x) \leq l(x) + c), \tag{1}$$

$$|\{x : l(x) = n,\ K(x) \geq n - m\}| \geq 2^n(1 - 1/2^m), \tag{2}$$

($|A|$ is the number of elements in A).

In [3.1.h] it was proved that

$$m - 2\log m \leq \log|\{x : K(x) \leq m\}| \leq m, \tag{3}$$

$$\log|\{x : K(x/m) \leq m\}| \asymp m. \tag{4}$$

$(F \leq G \Leftrightarrow (\exists c)(\forall x)(F(x) \leq G(x) + c),\ F \asymp G \Leftrightarrow F \leq G \wedge G \leq F,\ \log = \log_2)$

In this paper we consider some improvements for (3) and (4). Proofs are similar to the proofs for 1. (h) in [3], and they are based on Theorem 1.6 in [2].

(a) *Let*

$$A_n = \{x : l(x) = n,\ K(x) \geq n\}, \qquad A'_n = \{x : l(x) = n,\ K(x/n) \geq n\},$$
$$B_n = \{x : l(x) = n,\ K(x) \geq n - 1\}, \qquad B'_n = \{x : l(x) = n,\ K(x/n) \leq n - 1\},$$

Then

$$n - 2\log n \leq \log|A_n| \leq n, \log|A'_n| \asymp n, \tag{5}$$

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$$n - 2\log n \le \log|B_n| \le n, \log|B'_n| \asymp n. \tag{6}$$

We note that this is an improvement of (2), in case $m = 0$.

PROOF. We have immediately $|A_n| + |B_n| = 2^n$, $\log|A_n| \le n$, $\log|B_n| \le n$, and by [3, 1. g] $\min\{K(x) : l(x) = n\} \asymp K(n) \le l(n)$, which implies $(\forall n_0)(\exists n \ge n_0)(\exists x)(l(x) = n,\ K(x) < n)$, or $B_n \ne \emptyset$ for $n \ge n_0$. On the other hand $x \in B_n$ means $(\exists p_x)(l(p_x) \le n - 1,\ F_0^1(p_x) = x)$ and the number of programs p_x is at the most $2^0 + 2^1 + \ldots + 2^{n-1} = 2^n - 1 < 2^n$, which implies that $0 < |B_n| < 2^n$, $0 < |A_n| < 2^n$, or $A_n \ne \emptyset$ for $n \ge n_0$.

Let, for given $p = \bar{a}b$, the function $F(p)$ be defined in such a way that we first choose the set A of exactly b sequences y for which $K(y) \le a$, and then $x = F(p)$ is the first y such that $y \notin A$. Then for $a = n$, $b = |B_n|$, we have $x = F(p) \notin B_n$, and $n \le K(x) \le K_F(x) \le l(p) \le \log|B_n| + 2\log n$. Further, let $F'(\bar{a}b) = F(\bar{a}(2^a - b))$, and then $F'(\bar{n}|B_n|) = x \notin B_n$, which implies $n \le K(x) \le K_{F'}(x) \le l(\bar{n}|A_n|) \le \log|A_n| + 2\log n$.

In the case concerning B'_n, we can take $F(b, a) = F(\bar{a}b)$, and then $F(|B'_n|, n) = F(\bar{n}|B'_n|) = x \notin B'_n$, or $n \le K(x\|)\| \le K_F^-(x/n) \le l(|B'_n|) \asymp \log|B'_n| \le n$. The proof for $|A'_n|$ is similar.

(b) *Let $g(n)$ be a recursive increasing function, and $g(n) < n - 1$. Let*

$$A_n = \{x : l(x) = n,\ K(x) > g(n)\}, \quad A'_n = \{x : l(x) = n,\ K(x/n) > g(n)\},$$
$$B_n = \{x : l(x) = n,\ K(x) \le g(n)\}, \quad B'_n = \{x : l(x) = n,\ K(x/n) \le g(n)\}.$$

Then

$$\log|A_n| \asymp \log|A'_n| \asymp n, \tag{7}$$
$$g(n) - 2\log n \le \log|B_n| \le g(n),\ \log|B'_n| \asymp g(n). \tag{8}$$

PROOF. Let c_0 be such that $(\forall n)(\exists x)(l(x) = n,\ K(x) \le \log n + c_0)$, by [3, 1. g]. Let $g(n) - 2\log n \ge c_0$. Then $B_n \ne \emptyset$, and imitating the proof in (a), we can get (8). If $g(n) - 2\log n < c_0$, then (8) is trivial. Equality (7) is a direct consequence of (2), or simply $2^n \ge |A_n| = 2^n - |B_n| \ge 2^n - (2^{g(n)+1} - 1) \ge 2^n - 2^{n-1} = 2^{n-1}$, and $n \ge \log|A_n| \ge n - 1$.

(c) *Let*

$$D_n = \{x : l(x) \le n,\ K(x) > n\}, \qquad D'_n = \{x : l(x) \le n,\ K(x/n) > n\},$$
$$E_n = \{x : l(x) \le n,\ K(x) \le n\}, \qquad E'_n = \{x : l(x) \le n,\ K(x/n) \le n\},$$
$$c_n = \max\{K(x) - n : l(x) = n\}, \qquad c'_n = \max\{K(x/n) - n : l(x) = n\},$$
$$B_n = \{x : l(x) = n,\ K(x) = n + c_n\}, \quad B'_n = \{x : l(x) = n,\ K(x/n) = n + c'_n\},$$
$$F_n = \{x : l(x) = n,\ K(x) > n\}, \qquad F'_n = \{x : l(x) = n,\ K(x/n) > n\}.$$

Then

$$n - 2\log n \le \log|B_n| \le n,\ \log|B'_n| \asymp n, \tag{9}$$

$$n - 2\log n \leq \log|D_n| \leq n,\ \log|D'_n| \asymp n, \tag{10}$$

$$n - 2\log n \leq \log|E_n| \leq n,\ \log|E'_n| \asymp n, \tag{11}$$

and for infinitely many n, $c_n > 0$, *or* $F_n \neq \emptyset$, *and in that case*

$$n - 2\log n \leq \log|F_n| \leq n,\ \log|F'_n| \asymp n. \tag{10}$$

PROOF. By definition of c_n, and by (1), $0 \leq c_n \leq c$, and $B_n \neq \emptyset$, for all n. Similarly to the previous case we can prove (9). In order to prove (10), we shall prove that $D_n \neq \emptyset$. Let $H(n) = 2^n - 1$, and $h(n) = [\sqrt{n}]$ (integer part of $\sqrt{n}$). Then $K(H(H(h(n)))) \asymp K(h(n)) \leq l(n)/2$, but $l(H(H(h(n)))) = H(h(n)) > n$, which implies that for every $n \geq n_0$ there exists an x such that $K(x) < n$ and $l(x) > n$, which is equivalent to the existence of an y such that $l(y) \leq n$ and $K(y) > n$, which means $D_n \neq \emptyset$ for $n \geq n_0$. Then, it is easy to prove (10). Furthermore, $D_n \neq \emptyset$ for $n \geq n_0$, implies that for infinitely many n we have $F_n \neq \emptyset$, and then (12).

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