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# ON THE STABILITY OF THE FUNCTIONAL QUADRATIC ON $A$-ORTHOGONAL VECTORS

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**Abstract.** Let $X$ be a complex Hilbert space ($\dim X$ is at least three) and $A$ a bounded selfadjoint operator on $X$ ($\dim AX$ is neither 1 nor 2). In this paper we study a continuous functional $h$ on $X$ which is approximately quadratic on $A$-orthogonal vectors (i.e. $(\alpha_1)$ is satisfied provided that $(Ax, y) = 0$). We find that there exists a unique continuous functional $h_1$ (given by $(\alpha_2)$) which is quadratic on $A$-orthogonal vectors (i.e. $(\alpha_3)$ holds) and which is near $h$ (i.e. $(\alpha_4)$ holds).

In [1] the "approximately linear" mapping $f : E_1 \to E_2$ was considered, where $E_1$, $E_2$ are Banach spaces and a linear mapping $T : E_1 \to E_2$ which is near the mapping $f$ was found. In [2] we consider the mapping $\varphi : X \to X$, where $X$ is a complex Hilbert space which is "approximately additive" on $A$-orthogonal vectors, and we found the mapping $\varphi_1 : X \to X$ which is additive on $A$-orthogonal vectors and near the mapping $\varphi$. In this paper we consider the analogous problem for a functional which is "approximately square" on $A$-orthogonal vectors. For this functional the following theorem will be proved:

THEOREM. *Let $X$ be a complex Hilbert space $\dim X \geq 3$, $A : X \to X$ a bounded selfadjoint operator, $\dim AX \neq 1, 2$, $\theta \geq 0$ and $p \in [0, 2)$ real numbers and $h$ a continuous functional defined on $X$. If*

$$|h(x+y) + h(x-y) - 2h(x) - 2h(y)| \leq \theta[|(A_x, x)|^{p/2}] \tag{$\alpha_1$}$$

*whenever $(Ax, y) = 0$, then by*

$$h_1(x) = \lim_{n\to\infty} 2^{-2n} h(2^n x), \tag{$\alpha_2$}$$

*a continuous functional is defined on $X$ such that*

$$h_1(x+y) + h_1(x-y) = 2h_1(x) + 2h_1(y) \tag{$\alpha_3$}$$

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*whenever* $(Ax, y) = 0$. *Furthermore there exist a real number* $\varepsilon > 0$ *such that*

$$|h(x) - h_1(x)| \le |(Ax, x)|^{p/2} \cdot \varepsilon. \tag{$\alpha_4$}$$

PROOF. It is obvious that $h(0) = 0$.

1° Let $(Ax, x) = 0$ for some $x$ in $X (x \neq 0)$. Then it follows that

$$|h(x+x) + h(x-x) - 2h(x) - 2h(x)| \le 2\theta |(Ax, x)|^{p/2} = 0, \ \ |h(2x) - 4h(x)| = 0$$

respectively

$$h(2x)/2^2 = h(x) \tag{1}$$

Therefore, for each natural number $n$, taking into consideration that (1) holds $h(2^n x)/2^{2n} = h(x)$ and it is obvious that $\lim_{n\to\infty} h(2^n x)/2^{2n} = h(x)$.

2° Let $(Ax, x) \neq 0$ for some $x \in X$. Let us show first that there exists a $y \in X$ such that $(Ay, y) \neq 0$ and $(Ax, y) = 0$. Let $Y = \{y \mid y \in X, \ (y, Ax) = 0\}$ and let $(Ay, y) = 0$, for each $y \in Y$. From this for $y_1, y_2 \in Y$ we have $(A(y_1 + y_2), \ y_1 + y_2) = 0$, that is $(Ay_1, y_2) + (Ay_2, y_1) = 0$. If we replace $iy_2$ with $y_2$, we get $-i(Ay_1, y_2) + i(Ay_2, y_1) = 0$, and with the above equality we have $(Ay_2, y_1) = 0$ for every $y_1, y_2 \in Y$. Since for $y \in Y$ it follows that $Ay \perp Y$, $Ay = \alpha(y)Ax$, for each $y \in Y$. Since for each $z$ in $X$ can be written in the form $z = \beta x + y (x \notin Y, \ y \in Y)$, we have $Az = \beta Ax + Ay = (\beta + \alpha(y))Ax$, which contradicts the hypothesis that $\dim A(X) \neq 1$. Therefore there exists a vector $y \in X$, such that $y \perp Ax$ and $(Ay, y) \neq 0$. Without loss of generality we can assume that $(Ay, y) = \pm(Ax, x)$.

a) Let $(Ax, x) \neq 0$, $(Ax, y) = 0$, and furthermore let $(Ay, y) = (Ax, x)$. From this it follows that $(A(x+y), x-y) = 0$. Then we have that

$$\begin{aligned} &|h(x+y+x-y) + h(x+y-x+y) - 2h(x+y) - 2h(x-y)| \\ &\quad \le \theta[|(A(x+y), x+y)|^{p/2} + |(A(x-y), x-y)|^{p/2}] \end{aligned}$$

or $|h(2x) + h(2y) - 2h(x+y) - 2h(x-y)| \le \theta 2^{p/2+1} \cdot |(Ax, x)|^{p/2}$. Moreover

$$\begin{aligned} |[h(2x) - 4h(x)] + [h(2y) - 4h(y)]| &= |[h(2x) + h(2y) - 2h(x+y) - 2h(x-y)] \\ &\quad + [2h(x+y) + 2h(x-y) - 4h(x) - 4h(y)]| \le |h(2x) + h(2y) - 2h(x+y) \\ &\quad - 2h(x-y)| + 2|h(x+y) + h(x-y) - 2h(x) - 2h(y)| \\ &\quad \le \theta 2^{p/2+1} \cdot |(Ax, x)|^{p/2} + 2^2\theta |(Ax, x)|^{p/2} = 2\theta |(Ax, x)|^{p/2}(2 + 2^{p/2}). \end{aligned}$$

From that dividing by $2^2$ we obtain

$$|[h(2x)/2^2 - h(x)] + [h(2y)/2^2 - h(y)]| \le \theta |(Ax, x)|^{p/2}(1 + 2^{p/2-1}). \tag{2}$$

We check that the following inequality

$$\begin{aligned} &|[h(2^n x)/2^{2n} - h(x)] + [h(2^n y)/2^{2n} - h(y)]| \\ &\quad \le \theta |(Ax, x)|^{p/2} \cdot (1 + 2^{p/2-1}) \cdot \sum_{k=0}^{n} 2^{k(p-2)} \\ &\quad \le |(Ax, x)|^{p/2} \cdot (1 + 2^{p/2-1}) 2^2/(2^2 - 2^p) \end{aligned} \tag{*}$$

holds for each natural number $n$. It is obvious that $(^*)$ holds for $n=0$. Let $(^*)$ hold for $n=0,1,\dots,k-1$. We prove that $(^*)$ holds for $n=k$.

$$\begin{aligned}
&|[h(2^k x)/2^{2k}-h(x)]-h(x)]\\
&\quad+[h(2^k y)/2^{2k}-h(y)]|=|[h(2^{k-1}\cdot 2x)/2^{2(k-1)+2}-h(2x)/2^2]\\
&\quad+[h(2^{k-1}\cdot 2y)/2^{2(k-1)+2}-h(2y)/2^2]+[h(2x)/2^2-h(x)]+[h(2y)/2^2-h(y)]|\\
&\quad\le 1/2^2\cdot|[h(2^{k-1}\cdot 2x)/2^{2(k-1)}-h(2x)]+[h(2^{k-1}\cdot 2y)/2^{2(k-1)}-h(2y)]|\\
&\quad+|[h(2x)/2^2-h(x)]+[h(2y)/2^2-h(y)]|\le 1/2^2\cdot\theta|(A(2x),2x)|^{p/2}(1+2^{p/2-1})\\
&\quad\cdot 2^2/(2^2-2^p)+\theta|(Ax,x)|^{p/2}(1+2^{p/2-1})=\theta|(Ax,x)|^{p/2}(1+2^{p-2}\cdot 2^2/(2^2-2^p))\\
&\quad=\theta|(Ax,x)|^{p/2}\cdot(1+2^{p/2-1})2^2/(2^2-2^p).
\end{aligned}$$

Thus $(^*)$ holds for $n=k$; hence it holds also for each natural $n$. We prove that the sequence $\{h(2^n x)/2^{2n}+h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ converges for the $x,y\in X$ above. For $m>n>0$ we have

$$\begin{aligned}
&|[h(2^m x)/2^{2m}+h(2^m y)/2^{2m}]-[h(2^n x)/2^{2n}+h(2^n y)/2^{2n}]|=1/2^{2n}\\
&\cdot|[h(2^{m-n}\cdot 2^n x)/2^{2(m-n)}-h(2^n x)]+[h(2^{m-n}\cdot 2^n y)/2^{2(m-n)}-h(2^n y)]|\\
&\le 1/2^{2n}\cdot\theta|(A(2^n x),2^n x)|^{p/2}(1+2^{p/2-1})2^2/(2^2-2^p)=\theta|(Ax,x)|^{p/2}(1+\\
&2^{p/2-1})\cdot 2^2/(2^2-2^p)\cdot 2^{n(p-2)}.
\end{aligned}$$

Therefore, the sequence $\{h(2^n x)/2^{2n}+h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ is a Cauchy sequence; so it converges for the $x,y\in X$ above.

Let us show first that there exists a $z\in X$ such that $(Ax,z)=0$, $(Ay,z)=0$ and $(Ay,y)=(Az,z)=\pm(Ax,x)$. Let $P$ be a projection of $X$ onto $Y$ parallel to $Ax$. Then $Ay=\alpha(y)Ax+PAy$, and therefore $(PAy,y)=(Ay,y)$. (The last equality holds if $y$ is replaced by any other $u\in Y$.) Let $Z=\{z\mid z \text{ in } Y,\ (PAy,z)=0\}$. If $(PAz,z)=0$, for all $z\in Z$, then $(Az,z)=0$, for all $z\in Z$. Hence, $(Az,z_1)=0$, for all $z,z_1\in Z$. This means that $Az\perp Z$, so that $PAz=\alpha(z)\cdot PAy$, for all $z\in Z$. It is easy to check that $x$ is not in $Y$ and $y$ is not in $Z$ and that $x$ and $y$ are linearly independent. So every $y'\in Y$ can be written in the form $y'=\alpha x+\beta y+z$, and moreover

$$\begin{aligned}
PAy'&=\alpha PAx+\beta PAy+PAz\\
&=\beta PAy+PAz=\beta PAy+\alpha(z)PAy=(\beta+\alpha(z))PAy,
\end{aligned}$$

for all $y'\in Y$.

Let $u\in X$. Then

$$Au=\alpha(u)Ax+PAu \tag{$\delta$}$$

Since $u=\alpha_1 x+\beta_1 y+z$, this amounts to $PAu=\alpha_1 PAx+\beta_1 PAy+PAz=\beta_1 PAy+PAz$. Taking into consideration what was already proved, i.e. that $PAz=\beta(z)\cdot PAy$, we have $PAu=\beta(u)\cdot PAy$. Thus the relation $(\delta)$ becomes $Au=\alpha(u)\cdot Ax+\beta(u)\cdot PAy$, for all $u\in X$ which contradicts the hypothesis

$\dim A(X) \neq 1, 2$. Therefore, there exists a $z' \in Z$ such that $(PAz', z') \neq 0$. But $z \in Z$ can be chosen such that $(PAz, z) = (PAy, y)$, i.e. $(Az, z) = (Ay, y)$. Besides $(Ay, z) = 0$ and $(Ax, z) = 0$. If $(Ax, z) = 0$, $(Ay, z) = 0$ and $(Ay, y) = (Az, z) = (Ax, x)$, then from what was proved earlier it follows that the sequences

$$\{h(2^n x)/2^{2n} + h(2^n z)/2^{2n}\}_{n=0}^{\infty} \tag{$\Delta_1$}$$

$$\{h(2^n y)/2^{2n} + h(2^n z)/2^{2n}\}_{n=0}^{\infty} \tag{$\Delta_2$}$$

converge. If we subtract $(\Delta_2)$ from $(\Delta_1)$, we conclude that the sequence: $\{h(2^n x)/2^{2n} - h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ is convergent, which together with the fact the sequence $\{h(2^n x)/2^{2n} + h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ is convergent, implies that the sequence $\{h(2^n x)/2^{2n}\}_{n=0}^{\infty}$ is convergent for the $x \in X$ above.

b) Let $(Ax, x) \neq 0$ and $(Ax, y) = 0$, and furthermore let $(Ay, y) = -(Ax, x)$. From this it follows that $(A(x \pm y).\ x \pm y) = 0$. Then using 1° we have $h[2^n(x \pm y)]/2^{2n} = h(x \pm y)$ for each natural number $n$; hence, the sequences $\{h[2^n(x \pm y)]/2^{2n} = h(x \pm y)\}_{n=0}^{\infty}$ are convergent for the $x, y \in X$ above.

Since for every $x, y \in X$, for which $(Ax, y) = 0$ and $(Ay, y) = -(Ax, x)$ and all $n = 0, 1, 2, \ldots$ we have

$$\begin{aligned} &|h[2^n(x+y)]/2^{2n} + h[2^n(x-y)]/2^{2n} - 2h(2^n x)/2^{2n} - 2h(2^n y)/2^{2n}| \\ &\qquad \leq 2\theta 2^{n(p-2)} |(Ax, x)|^{p/2} \end{aligned}$$

we conclude that the sequence $\{h(2^n x)/2^{2n} + h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ converges for the $x$, $y \in X$ above. Similarly we conclude that the sequence $\{h(2^n x)/2^{2n} + h(2^n z)/2^{2n}\}_{n=0}^{\infty}$ is convergent, for each $x, z \in X$, for which $(Ax, z) = 0$ and $(Az, z) = -(Ax, x)$ (Before we showed that such a $z$ exists). Also, using a), for that $z$ it holds that $(Ay, z) = 0$ and $(Ay, y) = (Az, z)$, and we concluded that the sequence $\{h(2^n y)/2^{2n} + h(2^n z)/2^{2n}\}_{n=0}^{\infty}$ is convergent. From the fact that the sequences $\{h(2^n x)/2^n + h(2^n z)/2^{2n}\}_{n=0}^{\infty}$ and $\{h(2^n y)/2^{2n} + h(2^n z)/2^{2n}\}_{n=0}^{\infty}$ are convergent, we conclude that the sequence $\{h(2^n x)/2^n - h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ is convergent, which together with the fact that the sequence $\{h(2^n x)/2^{2n} + h(2^n y)/2^{2n}\}_{n=0}^{\infty}$ is convergent, implies that the sequence $\{h(2^n x)/2^{2n}\}_{n=0}^{\infty}$ is convergent for the $x \in X$ above. From that we conclude that for any $x \in X$, $\lim_{n\to\infty} h(2^n x)/2^{2n}$ exists. Let $h_1(x) = \lim_{n\to\infty} h(2^n x)/2^{2n}$, for all $x \in X$. Let us prove that for the functional $h_1$ we have: $h_1(x+y) + h_1(x-y) = 2h_1(x) + 2h_1(y)$, for all $x, y \in X$ for which is $(Ax, y) = 0$. For every $x, y \in X$ for which $(Ax, y) = 0$ and all $n = 0, 1, 2, \ldots$ it holds that

$$\begin{aligned} &|h[2^n(x+y)]/2^{2n} + h[2^n(x-y)]/2^{2n} - 2h(2^n x)/2^{2n} - 2h(2^n y)/2^{2n}| \\ &\quad \leq 2^{n(p-2)}\theta[|(Ax, x)|^{p/2} + |(Ay, y)|^{p/2}] \end{aligned}$$

Letting $n \to \infty$ we have $h_1(x+y) + h_1(x-y) = 2h_1(x) + 2h_1(y)$. Let $x$, $y$, $z$ be such that $(Ax, y) = 0$, $(Ax, z) = 0$, $(Ay, z) = 0$ and $\pm(Ax, x) = (Ay, y) = (Az, z)$. Then for all $n = 0, 1, 2, \ldots$ using the relation $(^*)$ we have

$|[\lim_{n\to\infty} h(2^n x)/2^{2n} - h(x)] + [\lim_{n\to\infty} h(2^n y)/2^{2n} - h(y)]| \le |(Ax,x)|^{p/2} \cdot \varepsilon_1$ where $\varepsilon_1 = \theta(1 + 2^{p/2-1}) \cdot 2^2/(2^2 - 2^p)$ or

$$\begin{aligned} &|[h_1(x) - h(x)] + [h_1(y) - h(y)]| \le \varepsilon_1 |(Ax,x)|^{p/2} \\ &|[h_1(x) + h_1(y)] - [h(x) + h(y)]| \le \varepsilon_1 |(Ax,x)|^{p/2}. \end{aligned} \tag{$\delta_1$}$$

By analogy we can prove that

$$|[h_1(x) + h_1(z)] - [h(x) + h(z)]| \le \varepsilon_1 |(Ax,x)|^{p/2} \tag{$\delta_2$}$$

$$|[h_1(y) + h_1(z)] - [h(y) + h(z)]| \le \varepsilon_1 |(Ax,x)|^{p/2} \tag{$\delta_3$}$$

Taking into consideration the relations $(\delta_1)$, $(\delta_2)$ and $(\delta_3)$ it is easy to check that: $|h(x) - h_1(x)| \le \varepsilon |(Ax,x)A^{p/2}$, for all $x$ in $X$ where $\varepsilon_1 = 3/2 \cdot \theta(1 + 2^{p/2-1} \cdot 2^2/(2^2 - 2^p)$. Before proving a continuity functional, we introduce the following: for $x, y \in X$ for which $(Ax,y) = 0$ and $(Ay,y) = \pm(Ax,x)$ let $\hat{h}_n^{\pm}\{x,y\} = h(2^n x)/2^{2n} \pm h(2^n y(/2^{2n}$. For $m > n > 0$ we have

$$\begin{aligned} &|\hat{h}_m^{+}\{x,y\} - \hat{h}_n^{+}\{x,y\}| = |h(2^m x)/2^{2m} + h(2^m y)/2^{2m} - h(2^n x)/2^{2n} - h(2^n y)/2^{2n}| \\ &= |[h(2^m x)/2^{2m} - h(2^n x)/2^{2n}] + [h(2^m y)/2^{2m} - h(2^n y)/2^{2n}]| = 1/2^{2n} \\ &\cdot |[h(2^{m-n} \cdot 2^n x)/2^{2(m-n)} - h(2^n x)] + [h(2^{m-n} \cdot 2^n y)/2^{2(m-n)} - h(2^n y)]| \\ &\le \theta/2^{2n} \cdot |(A(2^n x), 2^n x)|^{p/2} (1 + 2^{p/2-1}) \cdot 2^2/(2^2 - 2^p) \\ &= 2^2/(2^2 - 2^p) \cdot \theta |(Ax,x)|^{p/2} \cdot (1 + 2^{p/2-1}) \cdot 2^{n(p-2)}. \end{aligned}$$

It means that

$$|\hat{h}_m^{+}\{x,y\} - \hat{h}_n^{+}\{x,y\}| \le 2^2\theta/(2^2 - 2^p) \cdot |(Ax,x)|^{p/2} (1 + 2^{p/2-1}) 2^{n(p-2)} \tag{**}$$

A similar relation holds for the function $\hat{h}_n^{-}$. The domain functionals $\hat{h}_n^{pm}$ are set as follows:

$$\mathcal{D}^{\pm} \equiv \{\{x,y\} \mid (Ax,y) = 0,\ \ (Ay,y) = \pm(Ax,x), x,y \in X\} \subset X \times X.$$

The functionals $\hat{h}_n^{\pm}$ are continuous on $\mathcal{D}^{\pm}$, and from $(^{**})$ it follows that $\hat{h}_n^{\pm}$ uniformly converges on $\mathcal{D}^{\pm} \cap S$, where $S$ is any sphere in $X \times X$, and the functionals

$$\hat{h}^{\pm}\{x,y\} = \lim_{n\to\infty} [h(2^n x)/2^{2n} \pm h(2^n y)/2^{2n}] \tag{3}$$

are continuous on $\mathcal{D}^{\pm}$.

We now prove the continuity of the functional $h_1$. Let the sequence $\{x_n\}_{n=1}^{\infty}$ (in $X$) converge to $x_0$ (in $X$).

A) Suppose that $(Ax_0, x_0) \ne 0$.

1° Let $y_0 \in X$ $(y_0 \ne 0)$ such that $(Ax_0, y_0) = 0$ and $(Ay_0, y_0) = (Ax_0, x_0)$. Then $\{x_0, y_0\} \in \mathcal{D}^{+}$. As before, we can find a $z_0$, such that $(Ax_0, z) = 0$, $(Ay_0, z_0) = 0$ and $(Ay_0, y_0) = (Az_0, z_0) = (Ax_0, x_0)$. Therefore $\{x_0, z_0\} \in \mathcal{D}^{+}$, and $\{y_0, z_0\} \in$

$\mathcal{D}^+$. Since $(Ax_n, x_n) \to (Ax_0, x_0) \neq 0$ begins from any number $n$, $(Ax_n, x_n) \neq 0$, and we can begin from that $n$, we make a sequence

$$\tilde{y}_n = y_0 - x_n/(Ax_n, x_n) \cdot (Ay_0, x_n) \quad (n = 1, 2, \dots) \tag{4}$$

Obviously, $\tilde{y}_n \to y_0$ $(n \to \infty)$ and $(Ax_n, \tilde{y}_n) = 0$ $(n = 1, 2, \dots)$. From that we obtain

$$(A\tilde{y}_n, \tilde{y}_n) = (A\tilde{y}_n, y_0) = (\tilde{y}_n, Ay_0) = (y_0, Ay_0) - (Ay_0, x_n)/(Ax_n, x_n) \cdot (x_n, Ay_0) \tag{5}$$

For sufficiently large $n$, the difference on the right-hand side of this equality shall be different from zero, and we can begin from this $n$ and make the sequence

$$y_n = \sqrt{(Ax_n, x_n)/(A\tilde{y}_n, \tilde{y}_n} \cdot \tilde{y}_n \qquad (n = 1, 2, \dots) \tag{$\alpha$}$$

which has the following characteristics:

$$(Ax_n, y_n) = 0,\ (Ax_n, x_n) = (Ay_n, y) \text{ and } y_n \to y_0 \quad (n \to \infty).$$

Therefore, the pair $\{x_n, y_n\}$ are in $\mathcal{D}^+$. Since $(Ay_0, y_0) \neq 0$ and $(Ay_n, y_n) \to (Ay_0, y_0) \neq 0$ $(n \to \infty)$, we can make the sequence

$$\tilde{z}_n = z_0 - y_n/(Ay_n, y_n) \cdot (Az_0, y_n) - x_n/(Ax_n, x_n) \cdot (Az_0, x_n) \quad (n = 1, 2, \dots). \tag{6}$$

Hence $\tilde{z}_n \to z_0$ $(n \to \infty)$ and

$$\begin{aligned}
(Ay_n, \tilde{z}_n) &= (Ay_n, z_0 - y_n/(Ay_n, y_n) \cdot (Az_0, y_n) - x_n/(Ax_n, x_n) \cdot (Az_0, x_n)) \\
&= (Ay_n, z_0) - (Ay_n, z_0)(Ay_n, y_n)/(Ay_n, y_n) - (Ax_n, z_0)/(Ax_n, x_n) \cdot (Ay_n, x_n) \\
&= (Ay_n, z_0) - (Ay_n, z_0) = 0.
\end{aligned}$$

Therefore $(Ay_n, \tilde{z}_n) = 0$ $(n = 1, 2, \dots)$. Besides that, we have

$$\begin{aligned}
(A\tilde{z}_n, \tilde{z}_n) = (A\tilde{z}_n, z_0) = (z_0, Az_0) &- (Az_0, y_n)/(Ay_n, y_n) \cdot (y_n, Az_0) \\
&- (x_n, Az_0)/(Ax_n), x_n) \cdot (Az_0, x_n).
\end{aligned} \tag{7}$$

For a sufficiently large $n$ the difference on the right hand side of this equality shall be different from zero, and we can begin from this $n$, and make the sequence

$$z_n = \sqrt{(Ay_n, y_n)/(A\tilde{z}_n, \tilde{z}_n)} \cdot \tilde{z}_n \quad (n = 1, 2, \dots) \tag{$\beta$}$$

which has the following characteristics: $(Ay_n, z_n) = 0$, $(Ay_n, y_n) = (Az_n, z_n)$ and $z_n \to z_0$ $(n \to \infty)$. Therefore, the pairs $\{y_n, z_n\}$ are in $\mathcal{D}^+$. We now show that the pairs $\{x_n, z_n\}$ are in $\mathcal{D}^+$:

$$\begin{aligned}
(Ax_n, z_n) = \sqrt{(Ay_n, y_n)/(A\tilde{z}_n, \tilde{z}_n)} \cdot (Ax_n, \tilde{z}_n) &= \sqrt{(Ay_n, y_n)/(A\tilde{z}_n, \tilde{z}_n)} \cdot (Ax_n, z_0 \\
&- y_n/(Ay_n, y_n) \cdot (Az_0, y_n) - x_n/(Ax_n, x_n) \cdot (Az_0, x_n)) \\
&= \sqrt{(Ay_n, y_n)/(A\tilde{z}_n, \tilde{z}_n)}[(Ax_n, z_0) - (Ay_n, z_0)(Ax_n, y_n)/(Ay_n, y_n) \\
&- (Ax_n, z_0)(Ax_n, x_n)/(Ax_n, x_n)] = 0.
\end{aligned} \tag{***}$$

Therefore $(Ax_n, z_n) = 0$ and $(Ax_n, x_n) = (Ay_n, y_n) = (Az_n, z_n)$. Hence the pairs $\{x_n, z_n\}$ are in $\mathcal{D}^+$ $(n = 1, 2, \dots)$ too.

From these considerations we obtain $x_n \to x_0$, $y_n \to y_0$, $z_n, z_0$, $(n \to \infty)$.

Since the functional $h^+\{x, y\} = \lim_{n\to\infty}[h(2^n x)/2^{2n} + h(2^n y)/2^{2n}] = h_1(x) + h(y)$ is continuous on $\mathcal{D}^+$, we have

$$h^+\{x_n, y_n\} \to h^+\{x_0, y_0\} = h_1(x_0) + h_1(y_0);\ h^+\{y_n, z_n\} \to h^+\{y_0, z_0\}$$
$$= h_1(y_0) + h_1(z_0) h^+\{x_n, z_n\} \to h^+\{x_0, z_0\} = h_1(x_0) + h_1(z_0).$$

Subtract the second equation from the first to obtain $h^+\{x_0, y_0\} - h^+\{y_0, z_0\} = h_1(x_0) - h_1(z_0)$. By adding this equation to the third we obtain $h^+\{x_0, y_0\} - h^+\{y_0, z_0\} + h^+\{x_0, z\} = 2h_1(x_0)$. Therefore from $x_n \to x_0$ $(n \to \infty)$ it follows that

$$2h_1(x_n) = h^+\{x_n, y_n\} - h^+\{y_n, z_n\} + h^+\{x_n, z_n\} \to h^+\{x_0, y_0\} - h^+\{y_0, z_0\}$$
$$+ h^+\{x_0, z_0\} = 2h_1(x_0)$$

i.e.
$$h_1(x_n) \to h_1(x_0).$$

2° We proceed analogously and for the case $(Ax_0, y_0) = 0$ and $(Ay_0, y_0) = -(Ax_0, x_0)$ from $x_n \to x_0$ it follows that $h_1(x_n) \to h_1(x_0)$ $(n \to \infty)$.

B) Suppose that $(Ax_0, x_0) = 0$.

1° Let $Ax_n = 0$ $(n = 1, 2, \dots)$. Then we put $y_n = 0$ $(n = 1, 2, \dots)$. Then $(Ax_n, y_n) = 0$, $(Ax_n, x_n) = (Ay_n, y_n) = 0$ $(n = 1, 2, \dots)$. Therefore, the pairs $\{x_n, y_n\}$ are in $\mathcal{D}^\pm$ $(n = 1, 2, \dots)$, and we obtain $h^+\{x_n, y_n\} = h_1(x_n) + h_1(y_n)$, $h^-\{x_n, y_n\} = h_1(x_n) - h_1(y_n)$ or $2h_1(x_n) = h^+\{x_n, y_n\} + h^-\{x_n, y_n\}$. Since the functionals $h^+$ and $h^-$ are continuous, we have

$$2h_1(x_n) = h^+\{x_n, y_n\} + h^-\{x_n, y_n\} \to h^+\{x_0, y_0\} + h^-\{x_0, y_0\} = 2h_1(x_0).$$

Therefore from $x_n \to x_0$ it follows that $h_1(x_n) \to h_1(x_0)$ $(n \to \infty)$.

2° Suppose now that $Ax_n \neq 0$ $(n = 1, 2, \dots)$ and let $h_1(x) \not\to h_1(x_0)$ $(n \to \infty)$ i.e. let there be an $\varepsilon_0 > 0$ and let for the subsequence $\{x_{n_k}\}_{k=1}^\infty$ of the sequence $\{x_n\}_{n=1}^\infty$ the following hold

$$|h_1(x_{n_k}) - h_1(x_0)| > \varepsilon_0 \tag{8}$$

Put $x_{n_k} = \bar{x}_k$ $(k = 1, 2, \dots)$; then $|h_1(\bar{x}_k) - h_1(x_0)| > \varepsilon_0$. We construct the sequence $\{z_k\}_{k=1}^\infty = \{A\bar{x}_k/\|A\bar{x}_k\|\}_{k=1}^\infty$ $(\|z_k\| = 1,\ k = 1, 2, \dots)$. Since the sequence $\{z_k\}_{k=1}^\infty$ is bounded we can separate from it the subsequence $\{z_{k_p}\}_{p=1}^\infty$ which converges weakly to any $z_0$. Let $y_0 \neq 0$, $y_0 \perp z_0$ and $(Ay_0, y_0) \neq 0$. Then $(z_{k_p}, y_0) = (A\bar{x}_{k_p}/\|A\bar{x}_{k_p}\|, y_0) \to 0$ $(p \to \infty)$. We construct the sequence $\{y_p\}_{p=1}^\infty$, $y_p = y_0 - (A\bar{x}_{k_p}, y_0)/\|A\bar{x}_{k_p}\|^2 \cdot A\bar{x}_{k_p}$ $(p = 1, 2, \dots)$. Obviously $y_p \to y_0$ and $(A\bar{x}_{k_p}, y_0) \to (Ax_0, y_0) = 0 (p \to \infty)$. We shall show that $(A(x_0 \pm y_0), x_0 \pm y_0) \neq 0$.

From $(A(\bar{x}_{k_p} \pm y_p), \bar{x}_{k_p} \pm y_p) = (A\bar{x}_{k_p}, \bar{x}_{k_p}) \pm (A\bar{x}_{k_p}, y_p) \pm (Ay_p, y_p)$ letting $p$ tend to infinity we get

$$(A(\bar{x}_{k_p} \pm y_p), \bar{x}_{k_p} \pm y_p) \to (A(x_0 \pm y_0), x_0 \pm y_0) = (Ay_0, y_0) \neq 0.$$

Therefore we have $(Ax_0 \pm y_0), x_0 \pm y_0) = (Ay_0, y_0) \neq 0$. Then using $A$), the functional $h_1$ is continuous at the points $x_0 \pm y_0$ and $y_0$.

Since $(A\bar{x}_{k_p}, y_p) = 0$ $(p = 1, 2, \dots)$ and taking into consideration that $h_1$ is a quadratic functional on $A$-orthogonal vectors, we have $h_1(\bar{x}_{k_p} + y_p) + h_1(\bar{x}_{k_p} - y_p) = 2h_1(\bar{x}_{k_p}) + 2h_1(y_p)$ or $2h_1(\bar{x}_{k_p}) = h_1(\bar{x}_{k_p} + y_p) + h_1(\bar{x}_{k_p} - y_p) - 2h_1(y_p)$. If $p \to \infty$, we get $2h_1(\bar{x}_{k_p}) \to h_1(x_0 + y_0) + h_1(x_0 - y_0) - 2h_1(y_0) = 2h_1(x_0)$ which contradict (8). From $1^\circ$ and $2^\circ$ it now follows that in case $(Ax_0, x_0) = 0$, $x_n \to x_0$, $h_1(x_n) \to h_1(x_0)$ $(n \to \infty)$. This together with $A$) shows that $h_1$ is a continuous functional. Let us prove that the functional $h_1$ is unique. For that proof we shall use the fact that the quadratic functional is "square homogeneous" i.e. $h_1(ax) = |a|^2 h_1(x)$, for all $x \in X$ and $a$ is a complex number. Suppose that there is a continuous functional $h_2 \neq h_1$ for which $h_2(x + y) + h_2(x - y) = 2h_2(x) + 2h_2(y)$, for all $x, y \in X$ for which $(Ax, y) = 0$ and $|h(x) - h_2(x)| \leq \tilde{\varepsilon} \cdot |(Ax, x)|^{p/2}$, for all $x \in X$. Then $|h_1(x) - h_2(x)| \leq |h_1(x) - h(x)| + |h(x) - h_2(x)| \leq (\varepsilon + \tilde{\varepsilon})|(Ax, x)|^{p/2}$, for all $x \in X$, or

$$\begin{aligned} |h_1(x) - h_2(x)| &= |1/n^2 \cdot h_1(nx) - 1/n^2 \cdot h_2(nx)| = 1/n^2 \cdot |h_1(nx) - h_2(nx)| \\ &\leq 1/n^2 \cdot (\varepsilon + \tilde{\varepsilon}) n^p |(Ax, x)|^{p/2} = (\varepsilon + \tilde{\varepsilon}) n^{p-2} |(Ax, x)|^{p/2}. \end{aligned}$$

From this $\lim_{n \to \infty} |h_1(x) - h_2(x)| = 0$, for all $x \in X$. This proves the theorem.

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