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ON APPROXIMATION BY MODIFIED BERNSTEIN POLYNOMIALS

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Abstract. We study some theorems on the approximation of the r-th derivative of a given function f by corresponding r-th derivative of the modified Bernstein polynomials.

1. Introduction. For a function f defined on the closed interval $[0,1]$ of the real x-axis, the expression

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(k/n), \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, is called the Bernstein polynomial of order n of the function f.

Recently Derriennic [**1**] studied a new kind of positive linear operators $\{L_n\}$ for a Lebesque integrable function $f \in L^1[0,1]$, defined as

$$(L_n f)(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \tag{1.2}$$

which is a modification of the polynomials (1.1).

Following [**1**], for $r \leq n$, we get

$$(L_n^{(r)} f)(x) = \frac{(n+1)!n!}{(n-r)!(n+r)!} \sum{}^{*)} p_{n-r,k}(x) \int_0^1 p_{n+r,k+r}(t) \frac{d^r f(t)}{dt^r} dt \tag{1.3}$$

We will prove some theorems on the approximation of the r-th derivative of a function f by the corresponding operators $(L_n^{(r)})$.

2. In this section we prove the following lemma which will be useful in proving the theorems.

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$^{*)}\sum$ stands for $\sum\limits_{k=0}^{n-r}$.

LEMMA 2.1. For $r \leq n$, let

$$T_{n-r,m-r}(x) = \frac{m!}{(m-r)!} \sum p_{n-r,k}(x) \int_0^1 p_{n+r,k+r}(t)(t-x)^{m-r} dt. \tag{2.1}$$

Then one has the following relation

$$\begin{aligned}(n+m+2)\Big(\frac{m-r+1}{m+1}\Big)&T_{n-r,m-r+1}(x) \\ &= (m+1)(1-2x)T_{n-r,m-r}(x) + 2mx(1-x)T_{n-r,m-r-1}(x)+ \\ &+ x(1-x)T'_{n-r,m-r}(x),\end{aligned} \tag{2.2}$$

with $T_{n-r,0} = \frac{r!}{(n+r+1)}$. *In particular we get*

$$\frac{1}{(r+1)!} T_{n-r,1}(x) = \frac{(r+1)(1-2x)}{(n+r+2)(n+r+1)} \tag{2.3}$$

$$\begin{aligned}&\frac{2}{(r+2)!} T_{u-r,2}(x) = \\ &= \frac{(n+1)}{(n+r+3)(n+r+2)(n+r+1)} \Big\{ \frac{(r+1)(r+2)(1-2x)^2}{n+1} + 2x(1-x) \Big\}.\end{aligned} \tag{2.4}$$

Proof. Differentiating (2.1) w. r. t. x we get

$$\begin{aligned}x(1-x)T'_{n-r,m-r}(x) = \frac{m!}{(m-r)!} \sum_{k=0}^{n-r} p'_{n-r,k}(x)x(1-x) \int_0^1 p_{n+r,k+r}(t) \cdots \\ \cdots (t-x)^{m-r} dt - mx(1-x)T_{n-r,m-r-1}(x).\end{aligned}$$

Using the relation $x(1-x)p'_{n-r,k}(x) = \{k-(n-r)x\}p_{n-r,k}(x)$, we get after simplification

$$\begin{aligned}&x(1-x)T'_{n-r,m-r}(x) \\ &= \frac{m!}{(m-r)!} \sum p_{n-r,k}(x) \int_0^1 \{k-(n-r)t+(n-r)(t-x)\} \cdots \\ &\cdots p_{n+r,k+r}(t)(t-x)^{m-r} dt - mx(1-x)T_{n-r,m-r-1}(x), \\ &= \frac{m!}{(m-r)!} \sum p_{n-r,k}(x) \int_0^1 t(1-t)p'_{n+r,k+r}(t)(t-x)^{m-r} dt \cdots \\ &\cdots - r(1-2x)T_{n-r,m-r}(x) + \Big(\frac{m-r+1}{m+1}\Big)(n+r)T_{n-r,m-r+1}(x) - \\ &- mx(1-x)T_{n-r,m-r-1}(x).\end{aligned}$$

Again using $t(1-t) = -(t-x)^2 + (1-2x)(t-x) + x(1-x)$ and integrating the r. h. s. by parts, we get

$$\begin{aligned}&x(1-x)T'_{n-r,m-r}(x) \\ &= (m-r+2)\Big(\frac{m-r+1}{m+1}\Big)T_{n-r,m-r+1}(x) - (m-r+1)(1-2x)T_{n-r,m-r}(x) - \\ &- mx(1-x)T_{n-r,m-r-1}(x) - r(1-2x)T_{n-r,m-r}(x) + \\ &+ \Big(\frac{m-r+1}{m+1}\Big)(n+r)T_{n-r,m-r+1}(x) - mx(1-x)T_{n-r,m-r-1}(x)\end{aligned}$$

which on further simplification leads to the required result (2.2).

PROPOSITION 2.2. *For $r \le n$, we have*

(2.5)
$$L_n^{(r)}(t-x)(x) = \frac{n!(n+1)!}{(n-r)!(n-r+2)!}(1-2x)$$

(2.6)
$$L_n^{(r)}(t-x)^2(x) = \frac{(n+1)!^2}{(n-r)!(n-r+3)!}\left\{\frac{(r+1)(r+2)(1-2x)^2}{n+1} + 2x(1-x)\right\}$$

Proof. The proof follows easily from (1.3), (2.3) and (2.4).

PROPOSITION 2.3. *For $r \le n$, one has,*

(2.7)
$$L_n^{(r)}(t-x)^2(x) \le (r+2)/2(n+1), \quad (0 \le x \le 1).$$

Proof. We can easily show from (2.6) that,

$$L_n^{(r)}(t-x)^2(x) \le \frac{1}{n+r+2}\left\{\frac{(r+2)(r+1)(1-2x)^2}{n+1} + 2x(1-x)\right\}$$
$$\le \frac{(r+2)(r+1)}{(n+1)(n+r+2)} \le \frac{(r+2)}{2(n+1)}, \quad r \le n.$$

3. THEOREM 5.1. *If $f^{(r)}$ is bounded and integrable in $[0,1]$ and admits $(r+2)$-th derivative at a point $x \in [0,1]$, then*

(3.1)
$$\lim_{n\to\infty} n\{(L_n^{(r)}f)(x) - f^{(r)}(x)\} = (r+1)(1-2x)f^{(r+1)}(x) + x(1+x)f^{(r+2)}(x).$$

Proof. By the Taylor formula, we have

$$f^{(r)}(t) - f^{(r)}(x) = (t-x)f^{(r+1)}(x) + (t-x)^2/2 \cdot f^{(r+2)}(x) + (t-x)^2/2 \cdot \eta(t-x),$$

where $\eta(u) \to 0$ as $u \to 0$ and η is bounded in $[-x, 1-x]$ and integrable. Now, applying (1.3) to this and using the results (2.5) and (2.6), we get

(3.2)
$$(L_n^{(r)}f)(x) - f^{(r)}(x) = \left[(r+1)(1-2x)f^{(r+1)}(x) + \left\{\frac{(r+1)(r+2)(1-2x)^2}{2(n+1)}\right.\right.$$
$$\left.\left. +x(1-x)\right\}\frac{n+1}{n+r+3}f^{(r+2)}(x)\right]\frac{n!(n+1)!}{(n-r)!(n+r+2)!} + E_{n,e}(x)$$

where $E_{n,r}(x) = \dfrac{(n+1)!n!}{2(n-r)!(n+r)!}\sum p_{n-r,k}(x)\displaystyle\int_0^1 p_{n+r,k+r}(t)(t-x)^2\eta(t-x)dt.$

We shall show that $nE_{n,r}(x) \to 0$ as $n \to \infty$. Let $M = \sup\limits_{u\in[-x,1-x]} \mid \eta(u) \mid$ and let $\varepsilon < 0$ be arbitrary. Choose $\delta > 0$ such that $\mid \eta(u) \mid < \varepsilon$ when $\mid u \mid \le \delta$. So for all $t \in [0,1]$, we have $\mid \eta(t-x) \mid < \varepsilon + M(t-x)^2/\delta^2$. Clearly

$$nE_{n,r}(x) < \varepsilon n/2 \cdot L_e^{(r)}(t-x)^2(x) + Mn/2\delta^2 \cdot L_n^{(r)}(t-x)^4(x)$$
$$= \frac{\varepsilon n}{2}\frac{(n+1)!(n+1)!}{(n-r)!(n+r+3)!}\left\{\frac{(r+1)(r+2)(1-2x)^2}{n+1} + 2x(1-x)\right\} + \frac{M}{2\delta^2}O\left(\frac{1}{n}\right);$$

which show that $nE_{n,r}(x) \to 0$ as $n \to \infty$. Thus, as $n \to \infty$, we get the required result from (3.2). This completes the proof.

THEOREM 3.2. *Let* $f \in C^{(r+1)}[0,1]$ *and let* $w(f^{(r+1)};\cdot)$ *be the moduli of continuity of* $f^{(r+1)}$. *Then for* $n \geq r$, $(r = 0,1,2,\dots,)$, *we have*

$$\|L_n^{(r)}f - f^{(r)}\| \leq (r+1)/(n+r+2)\|f^{(r+1)}\| + 1/\sqrt{n}\{\sqrt{\lambda r} + \lambda r/2\}\cdots \cdots w(f^{(r+1)};1/\sqrt{n}), \tag{3.3}$$

where the norm is sup-norm over $[0,1]$, *and* $\lambda r = 1 + r/2$.

Proof. Following [**2**], we write

$$f^{(r)} - f^{(r)}(x) = (t-x)f^{(r+1)}(x) + \int_x^t \{f^{(r+1)}(y) - f^{(r+1)}(x)\}dy.$$

Now, applying (1.3) to the above and using the inequality

$$| f^{(r+1)}(y) - f^{(r+1)}(x) | \leq \{1 + | y - x | /\delta\} w(f^{(r+1)};\delta),$$

and the results (2.5) and (2.6), we get

$$\begin{aligned}
&| (L_n^{(r)}f)(x) - f^{(r)}(x) | \\
&\leq | f^{(r+1)}(x) || L_n^{(r)}(t-x)(x) | + w(f^{(r+1)};\delta) L_n^{(r)}\left[\left| \int_x^t 1 + | y - x | /\delta dy \right|\right](x), \\
&\leq | f^{(r+1)}(x) || L_n^{(r)}(t-x)(x) | + w(f^{(r+1)};\delta)\left\{ \sqrt{L_n^{(r)}(t-x)^2(x)} \right. \\
&\qquad\qquad \left. + L_n^{(r)}(t-x)^2(x)/2\delta \right\}.
\end{aligned}$$

Choosing $\delta = 1/\sqrt{n}$ and using the resilt (2.7), we get the required result (3.3). This completes the proof.

Remark. By putting $r = 0$ in (3.1), we get Theorem 2.5 of Derriennic [**1**].

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