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A TREE AXIOM[1]

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Abstract. In connection with my previous results from 1935, and results of other mathematicians (Tarski, Erdös, Hanf, Keisler, Baumgartger...) the following Tree (or Dendrity) Axiom is formulated: For any regular uncountable ordinal n there exists a tree An of height (rank) n such that $|X| < |n|$ for every level X as well as for every subchain X of An. In other words, the following assertion Dn holds: There exists a tree T such that for every regular ordinal $n > \omega_0$ the conditions (2:0), (2:1), (2:2) hold.

0. Studying since 1932 the well-known Suslin problem concerning simple ordered sets (chains), and transforming this problem into a problem concerning trees or ramified tables, I was lead to consider trees $(T, \leq)$ of height or rank $\gamma T = \omega_1$ such that $|X| < \aleph_1$, where X runs through the set of all rows of $(T, \leq)$ and the system of all subchains of $(T, \leq)$. I studied such trees irrespectively of their existence (cf. Kunen [1980, p. 69^{9-11}]): "Suslin trees were introduced by Kurepa (see, e.g. Kurepa [1936]), who showed that there is an ω_1-Suslin tree iff there is a Suslin line (see Theorem 5.13)". and Todorčević [1984] p. 246_{9-10}: "Aronszajn, Suslin and Kurepa trees were introduced by Kurepa [1935], [1937 a] and [1942]" (i.e. in Kurepa [1935 b, c], [1937 b] and [1942 a] of the present bibliography). I considered analogous situations for regular cardinals ω_β for every ordinal $\beta > 0$, in particular for any inaccessible ordinal $\beta > \omega_0$[2].

After many years let me announce the following

1. Tree (or Dendrity) Axiom: For any regular uncountable ordinal n there exists a tree An of height or rank n such that $|X| < |n|$; here X stands for any level or row of An or any subchain of the tree An.

2. Statement Dn: Let Dn or D(n) denote for any ordinal n the following assertion:

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[1]Presented 1985:10:02 at the 4-th International Conference on Topology and its Applications (Dubrovnik 1985:09:30:1–10:05); abstracts submitted 1985:09:21 to Matematički institut, Beograd, where 1985:10:11:5:18$^{\mathrm{h}}$ I delivered a one hour talk on the subject.

[2]Any infinite ordinal α is said to be inaccessible if it is a limit regular, thus $\mathrm{cf}\,\alpha = \alpha = \omega_\alpha$ and $\mathrm{cf}\,\omega_\alpha = \alpha$.

Dn There exists a tree T such that

(2:0) $\gamma T = n$,

(2:1) $|R_\alpha T| < |n|$ for any $\alpha < \gamma T$

(2:2) $|L| < |n|$ for any chain L in T.

2:3. Notation. For a tree T and every ordinal α we denoted by $R_\alpha T$ the set of all elements x of T such that the corresponding left interval $T(\cdot, x) := \{y \mid y \in T, y < x\}$ has the order-type α. The first α such that $R_\alpha T = \nu$ (vacuous or empty set) was called the rank or height of T and denoted by γT; $R_\alpha T$ was called the α-th level (row) of the tree T. In this way we obtained the well-determined fundamental disjoint partition $T = \cup R_\alpha T$ $(\alpha < \gamma T)$ of every tree T.

2:4. The pseudorank $\gamma' T$ of T is the greatest limit ordinal $\leq \gamma T$; if $\gamma T < \omega_0$, then $\gamma' T := 0$.

2:5. A classification of trees T. For any $\alpha < \gamma T$ let $m_\alpha T := |R_\alpha T|, mT := \sup m_\alpha T$, $(\alpha < \gamma T)$; T is said to be large if for some $\alpha < \gamma T$ one has $m_\alpha T \geq |\mathrm{cf}\, \gamma' T|$; T is said to be narrow if $mT < |\mathrm{cf}\, \gamma' T|$ and moreover, if $\mathrm{cf}\, \gamma' T = \omega_{\beta+1}$ then $mT < \aleph_\beta$. If T is neither narrow nor large, T is said to be ambiguous, nice or idoneus.

2:6. The falsity of $D\omega_0$ is the content of König's [1927] Infinity Lemma. Independently of this lemma I proved the falsity of $D\omega_\nu$ for any ω_ν cofinal with ω_0 – a particular case of the fact that every infinite narrow tree is equinumerous with one of its own subchains (v. Thèse [1935] p. 80 Th. 3^{bis}).

3. Genesis. In 1934 I gave the definition of decreasing trees of sets, T with an erroneous statement that T contains a chain intersecting every row of T (*lapsus calami*: it was not indicated that T should be "narrow").

3:1. In the next note [1934 d] the error was notified and it was indicated that Aronszajn gave me an exemple of an ambiguous ω_1-tree having no ω_1-branche. Aronszajn's construction was published in Kurepa [1935 b, c, p. 96][3].

3:2. At the same place was published my construction (found in 1934 after that of Aronszajn) and based uniquely on order considerations concerning the ordered set $(Q, \leq)$ of rational numbers. My starting point was the tree $\sigma(Q, \leq) := \sigma_0$ of all well-ordered bounded subsets of $(Q, \leq)$ ordered by the relation "to be an initial segment of"; σ_0 is a tree of nonattained rank ω_1 and its levels $R_\alpha \sigma_0$ are of power $\aleph_0$ for $\alpha < \omega_0$ and $2\aleph_0$ for $\omega \leq \alpha < \omega_1$.

3:3. I indicated that instead of $(Q, \leq)$ one could consider the Hausdorff set $H_0 := (l + m + n)^{\omega_0 *}$ = the system of all ω_0-sequences f of numbers of a given ordered set $\{l < m < n\}$, such that a right section of f equals the constant ω_0-sequence $m, m, \ldots$; the set H_0 is ordered by the principle of first differences. At p. 97_{12-10} I indicated: "Comme on a construit σ_0, S_0 à partir de H_0, on construit, à partir de H_β, les suites ramifiées σ_β, S_β. Nous ne le ferons pas". Thus $\sigma_\beta := \sigma(H_\beta, \leq)$, S_β is a $\omega_{\beta+1}$-tree $\subset \sigma_\beta$ of breadth $|\omega_\beta|$.

[3] I do not know why Aronszajn did not publish his construction and I am sorry that he didn't. Aronszajn and I met in Paris quite frequently in the years 1933–1935 and in 1937 (we prepared both our Theses with Fréchet). In particular he was a witness of the writing of my Thèse; he had a copy of the manuscript of my Thèse before its publication.

3:4. As a matter of fact the construction of S_0 was transferable *verbatim* for construction of $S_\beta \subset \sigma_\beta$ provided the ordinal β be such that $\sum_{\xi<\omega\beta} 3^{|\xi|} < 3^{|\omega\beta|}$; this situation occurs for every regular ordinal β, provided GCH is accepted.

3:5. Independently of GCH the existence of S_β, i.e. of ambiguous $A(\omega_{\beta+1})$-trees, was proved in Kurepa [1968 b] for every ordinal β; the construction is based on the following result, important by itself.

THEOREM. Kurepa [1935 b, c Th. 6 p. 89]. *Let T be a tree such that for every element t of T the set $R_0(t,\cdot)$ of all immediate successors of t in T is infinite; let this set be ordered totally in such a way that the ordering has no minimal element. We consider the corresponding natural ordering of $(T,\leq)$. Let a be any limit ordinal between 0 and γT. The sets $(\cdot,a)_T := \cup_{\xi<\alpha} R_\xi T$ and $R_\alpha T$ are multually dense (i.e. everywhere dense one into another) in this natural ordering $\leq n$ if and only if $\gamma[x]_{(T,\leq)} = \gamma(T,\leq)$ for every $x \in T$; one takes $[x]_T := \{y \in T,\ y$ is comparable to $x\}$.*

3:6. Case of inaccessible ordinals.

3:6:1. In Kurepa [1935 b, c; p. 100^{9-13}] one reads with corresponding italic characters: "Nous ne savons pas s'il existe une suite distinguée dont le rang serait un nombre inaccessible. Au contaire, quel que soit l'ordinal β, on peut démontrer l'existence d'une suite distinguée de rang $\omega_{\beta+1}$. On rencontrera à plusieurs reprises des suites distinguées". In the present terminology, we do not know whether there exists a tree $D(\omega_\xi)$ for inaccessible ω_ξ. On the contrary, for every ordinal β one can prove the existence of a $D(\omega_{\beta+1})$-tree. We shall enconter such trees a lot of times.

3:6:2. In Kurepa [1968:2, p. 153_{11}] one reads "The problem of the existence of A_ν for inaccesible $\nu > \omega_0$ remains open", A_ν denotes any ambiguous ω_ν-tree having no ω_ν-branche.

To make the terminology precise: every regular limit cardinal (ordinal) is said to be inaccessible; k is said to be strongly inaccessible if k is inaccessible and moreover for every $x < k$ one has $2^{|x|} < k$.

3:6:3. According to Erdös-Tarski [1961] $D(k) \Leftrightarrow k \nrightarrow (k)^2_2$ for every strongly inaccessible $[s,i]$ cardinal k.

3:6:4. According to Hanf [1964], for many s.i. cardinals one has $D(k)$. In particular, for the first s.i. ordinal i one has $D(i)$.

3:6:5. Now, we formulate that $D(k)$ holds for every inaccessible ordinal $k > \omega_0$ as well as for every $k = \omega_{\beta+1}$; irrespectively of whether the ordinal β is regular or singular; in particular our axiom yields $A\omega_{\omega+1}$.

3:6:6. In particular, our Tree Axiom expresses that for every strongly inaccessible cardinal $> \omega_0$ each of the statements (I)–(VII) in Theorem 29.6 p. 172 of the book Erdös-Hajnal-Rado-Maté [1984] holds.

4:1. What about maximum antichains in trees? In the period until 1937 it was not known whether the ambiguous or idoneous ω_1-trees, constructed in 1934 first by Aronszajn, and then by myself, contain an uncountable antichain; this question was indicated in my Thèse as an open problem (see Thèse, Introduction p. 3^{4-13},

p. 134_{7-1}); the problem was solved in 1937 affirmatively: from Glina I sent in 1937:08:23:1 a manuscript to Banach St. (Leopol) a solution of the problem. The paper was received in 1937:08:31 in Studia Mathematicae and published in 1940; I read no proof-sheets; I received the reprints in 1949:02:19 in Zagreb. The paper [1940] contains no typographical error; only, p. $25^{6,13}$ non $\rightarrow$ disjoints non.

4:2. The existence of $A(\omega_1)$ trees with no $\aleph_1$-antichain was considered by me as a postulate (see Thèse 1935 p. 134); this fact was confirmed as late as 1967–1971 by Jech, Tennenbaum, and Solovay.

5. Occurring of inaccessible numbers

5:0. In the Tree Axiom the role of inaccessible numbers is particularly important, and therefore the Tree Axiom can be considered mainly as a large cardinal axiom.

In this connection it is instructive to indicate the following facts.

5:1. In my doctoral dissertation the main alternative for infinite cardinals was the distinction between maximum and supremum and the question whether in given circumstances one has really a maximum and not only a supremum (under this main idea was formulated my ramification hypothesis). And in this respect the case of inaccessible numbers had a crucial role.

5:2. In my Thèse (p. 109) for any tree T I introduced a cardinal number $b'T$ which in the particular case when T is a decreasing tree of sets is just the supremum of $|D|$, D running through the system of all disjoint subsystems of $T^d{:}=\{X, X \in T$ or $X = Y \backslash Z$, where $Y, Z \in T$ and $Y \supset Z\}$. Then I proved the following

5:2:1. Theorem [Thèse p. 110, Théorème 3]. *Unless the tree T is of inaccessible rank, the supremum $b'T$ is attained.*

This has an obvious consequence concerning ordered chains $(E, \leq)$: Unless $\mathrm{cf}\, p_2E$ is inaccessible, the ordered space $(E, \leq)$ contains a disjoint system of open intervals of cardinality $p_2E{:}=$ the cellularity of $(E, \leq)$, i.e. $\sup_D |D|$, D being a disjoint system of open sets in $(E, \leq)$.

5:2:2. This fact is transferable to topological spaces, as was published without quotation of my result, in Erdös-Tarski [1943]. In this respect it is instructive to quote the starting lines of this paper (p. 315^{4-10}): "In this paper we shall be concerned with a certain particular problem for the general theory of sets, namely with the problem of the existence of familes of mutually exclusive sets with a maximal power. It will turn out – in a rather unexpected way – that the solution of these problems essentially involves the notion of the so-called "inaccessible numbers". In this connection we shall make some general remarks regarding inaccessible numbers in the last section of our paper".

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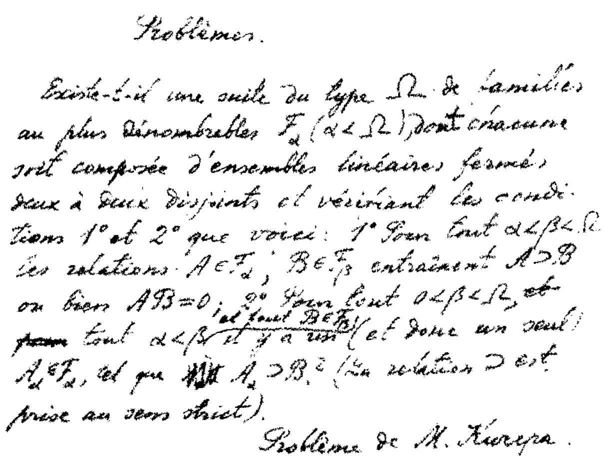
Notice: Ligne 135_5 remplacer par ceci: remarquable dû à moi et dont la démonstration correcte sur ma demande, est due à Aronszajn; je lui avais donné mon manuscript (démonstration), que, à Warszawa, j'avais donné, personnellement, pour Fundamenta Mathematicae, à Sierpiński, avant mon départ de Warszawa pour Paris (v. Introduction n^{e} 4)[4].



Problèmes.

Existe-t-il une suite du type Ω de familles au plus dénombrables F_α $(\alpha<\Omega)$, dont chacune soit composée d'ensembles linéaires fermés deux à deux disjoints et vérifiant les conditions 1° et 2° que voici: 1° Pour tout $\alpha<\beta<\Omega$ les relations $A\in F_\alpha$, $B\in F_\beta$ entraînent $A\supset B$ ou bien $AB=0$; 2° Pour tout $0<\beta<\Omega$, ~~et pour~~ tout $\alpha<\beta$ et tout $B\in F_\beta$ il y a un (et donc un seul) $A_\alpha\in F_\alpha$, tel que $A_\alpha\supset B$. (La relation $\supset$ est prise au sens strict).

Problème de M. Kurepa.

This text was written in 1937 by Sierpiński.

[4] At this opportunity, Sierpiński gave me back the formulation of my problem written in March 1937 by his own hand: I gave him my manuscript (positive answer to my problem), he gave me back the problem; v. Faccimile above.
Remark. The last phrase in the review in Zbl **20** (1940) 108 written by Blumberg (Columbia) should read: "This proposition has been announced by the author, who personally, before leaving Warszawa for Paris, end of April 1937, gave in Warszawa to Sierpiński his manuscript – a positive answer to his own problem, in order to be published in Fundamenta Mathematicae: in Paris the author discovered a gap in his proof; then he asked Aronszajn to bridge the gap; Aronszajn succeeded and found the proof displayed in the present paper (cf. Introduction, n° 4)".

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