PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 38 (52), 1985, pp. 21–30

BASES OF WEB CONFIGURATIONS

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Abstract. V. D. Belousov proved that a web configuration corresponds to a system of functional equation(s). A base of configuration (i.e. a minimal generating set of lines) is important in establishing the correspondence. The paper deals with the problems of existence of bases and algorithms for finding them.

Although the important relationships between webs (also called nets) and quasigroups have been known for some time [**1**], no significant advances for a general theory occurred until the work of Belousov on configurations [**2**], [**3**]. In his major work [**3**] in this area Belousov is primarily concerned with replacing a geometric constraint (configuration) on a web by a functional equations on quasigroups coordinatizing the web. This paper is written to give some further results in the field.

A system W of objects of two sorts, called points and lines, with an incidence relation between them is a k-web iff
– the set of all lines is partitioned into k disjoint classes
– two lines from different classes are incident to exactly one point
– in every line class there is exactly one line incident to a given point.

Usually the following axioms are added to exclude so called trivial webs.
– $k > 3$
– there are at least two points incident to every line.

A k-web is of order n if one line class (consequently all) has exactly n lines. In a k-web of order n there are exactly n^2 points and kn lines. Unless explicitely stated otherwise, we assume that both k and n are finite.

If S is a set of cardinality n there is a bijection between the set of points and $S \times S$, and consequently we can denote points of W by pairs (a, b), $a, b \in S$. Similarly the pair (p, c), $c \in S$ will be used to denote a line in the p-th line class $(0 < p \leq k)$. Finally, for $O < p \leq k$ we define operations $A_p : S \times S \to S$ by

$$A_p(a, b) = c \text{ iff the point denoted by } (a, b) \text{ is incident to the } p\text{-line denoted by } c.$$

AMS Subject Classification (1980): Primary 05B30, Secondary 20N05

The algebra $(S; A_1, \ldots, A_k)$ is called a coordinate system for W, (a, b) coordinates of the corresponding point (usually shortened to "point (a, b)") and (p, c) coordinates of the corresponding line ("line (p, c)" or "p-line c").

Any two operations A_p, A_q $(p \neq q)$ of a coordinate system of a web are orthogonal i.e. the system of equations:

$$A_p(x, y) = a, \quad A_q(x, y) = b, \quad (p \neq q)$$

has a unique solution for all $a, b \in S$.

Conversely any algebra $(S; A_1, \ldots, A_k)$ of k binary mutually orthogonal operations A_p $(0 < p \leq k)$ is a coordinate system of a k-web of order $|S|$ (see [**4**]).

Since the operations A_p, A_q $(p \neq q)$ are orthogonal, the function (A_p, A_q) : $S \times S \to S \times S$ defined by $(A_p, A_q)(x, y) = (A_p(x, y), A_q(x, y))$ is a bijection and consequently the operations $V_{pqr}(x, y) = A_r(A_p, A_q)^{-1}(x, y)(0 < p, q, r \leq k; p \neq q)$ are well defined. Moreover for $O < p,\ q, r \leq k;\ p \neq q$:

(1) $$V_{pqp}(x, y) = \pi_1(x, y) = x$$

(2) $$V_{pqq}(x, y) = \pi_2(x, y) = y$$

(3) $$V_{pqr} \text{ is a quasigroup iff } p \neq r \neq q$$

(4) $$V^*_{pqr} = V_{qpr}$$

(5) $$V^{-1}_{pqr} = V_{prq} \quad (q \neq r)$$

(6) $$V^{-1}_{pqr} = V_{rqp} \quad (p \neq r)$$

where V^*_{pqr}, V^{-1}_{pqr} and V^{-1}_{pqr}; are respectively dual, left and right inverse quasigroups of V_{pqr}.

(7) $$V_{pqr}(\pi_1(x, y), V_{pjq}(x, y)) = V_{pjr}(x, y)$$

(8) $$V_{pqr}(V_{iqp}(x, y), \pi_2(x, y)) = V_{iqr}(x, y)$$

(9) $$V_{pqr}(V_{ijp}(x, y), V_{ijq}(x, y)) = V_{ijr}(x, y)$$

Belousov introduced the algebra $(S; (V_{pqr})_{0<p,q\leq k, r\leq k; p\neq q})$ in [**3**] and called it a covering system of operations for $(S; A_1, \ldots, A_k)$. We will refer to it as a covering algebra.

Conversely, any algebra $(S; (V_{pqr})_{0<p,q\leq k, r\leq k; p\neq q})$ with properties (1) - (9) is a covering algebra i.e. there are mutually orthogonal binary operations $A_1, \ldots, A_k$ on S such that $V_{pqr}(x, y) = A_r(A_p, A_q)^{-1}(x, y)$ $(0 < p, q, r \leq k; p \neq q)$.

The geometric interpretation of the operations V_{pqr} is straightforward: $V_{pqr}(x, y)$ is the unique r-line incident to the unique point incident to both the p-line x and the q-line y.

Belousov also defines a configuration of a web W as a nonempty set M of points of W such that any point is collinear to some other point of M.

We shall call a line l of the web W a line of the configuration M if l is incident to at least two points of M.

The major results from [**3**] are about configurations which have the properties:

— every point of M is incident to exactly three lines of M.

— any two points of M are connected in M.

Two points of M are collinear in M if they are both incident to the same line of M. Connectedness in M is the transitive closure of this relation. We will assume that all configurations satisfy these two properties.

Sometimes we will consider configurations which in addition to previous properties have exactly two points incident to every line. Such a configuration is said to be (3, 2)-homogeneous.

In classical web theory the Thomsen, Reidemeister, Bol and hexagonal closure conditions are important [**1**]. Each of these closure conditions corresponds to a configuration (of the same name). The Thomsen, Reidemeister and hexagonal configurations are (3, 2)-homogeneous whereas the Bol configurations are not. Thus the class of (3, 2)-homogeneous configurations is an interesting orie to investigate.

Let us introduce constants for the lines of M. If l is a line of M with the coordinates (p, a) then let a be a constant for line l which is in S interpreted as a.

Admissible terms (relative to the configuration M) are defined by:

— if l is a line of M with coordinates (p, a) then a is admissible term for the line l

— if t_1, t_2 are admissible terms for lines l_1, l_2 with coordinates (p, a) and (q, b) respectively, and $V_{pqr}(a, b) = c$ then $V_{pqr}(t_1, t_2)$ is admissible term for the line with coordinates (r, c) provided that the line belongs to M

— a term t is admissible if it is admissible for some line of M.

As an immediate consequence we have:

LEMMA 1. *The value of any admissible term (relative to some configuration M) is a coordinate of a line of M.*

Admissible equality (relative to the configuration M) is the equality $t_1 = t_2$ such that both t_1 and t_2 are admissible terms for some line l of M.

Let B be the set of lines of a configuration M. Then $\langle B \rangle$ is the set of all lines from M generated by lines from B i.e. the second coordinate of every line of $\langle B \rangle$ is a value of some admissible term containing constants for lines from B only.

Lines of B are independent iff for every line l from B, the sets $\langle B \rangle$ and $\langle B \setminus \{l\} \rangle$ are different.

B is a base of a configuration M iff:

— B generates M,

— the lines of B are independent.

THEOREM 1. *Every configuration M of a finite web W has a base.*

Proof. The set of all lines of M is certainly a generating set but not an independent one. So we can choose a proper subseet which still generates M. If the new set of lines is not independent we can repeat the procedure. Since the web W is finite the procedure will terminate giving us a base of M. □

Although he does not state it, Theorem 1 is implicit in Belousov's work [**3**]. It can be easily generalized to:

Corollary 1. *Every finitely generated configuration of a (possibly infinite) web W has a base.*

A natural question to ask is whether any configuration of an arbitrary web has a base. The following example shows that there is an infinite configuration without a base so that the finiteness conditions in Theorem 1 and Corollary 1 are essential.

Example 1. We say that B is a base of a group G iff B is a minimal generating set of G.

Some groups have a base while others do not. For example any finitely generated group will have a base while any group of type p^∞ (Prüfer p-group) will not [**6**].

Let G be a group without a base. Let the set of points, of a web W be $P = G \times G$ and the set of lines of W be $L = \{(m,g)|\, m = 1,2 \text{ or } 3; g \in G\}$. The three line classes L_1, L_2, L_3 of W are obvious. The operations $V_{pqr}(0 < p,q,r \leq 3; p \neq q)$ are as follows:

$$V_{123}(g,h) = V_{213}(h,g) = gh$$
$$V_{l32}(g,h) = V_{312}(h,g) = g \setminus h = g^{-1}h$$
$$V_{231}(h,g) = V_{321}(g,h) = g/h = gh^{-1}$$

so we will say "product" instead of "admissible term" and write $\cdot$, $\setminus$ or $/$ instead of the corresponding operation V.

We will consider the (infinite) configuration $M = P$. Notice that all lines of W are lines of M.

Lemma 2. *If* $\langle \Gamma \rangle = L$ *then* Γ *is an infinite set of lines.*

Proof. Suppose $\langle \Gamma \rangle = L$ for some finite set Γ. Then $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ where $\gamma_m = (j_m, g_m)$ $(j_m \in \{1,2,3\}, g_m \in G, 0 < m \leq n)$. If $B = \{g_1, \dots, g_n\}$ then $\langle B \rangle \neq G$ and $\langle \Gamma \rangle \subset \{(j,g)|\,) < j \leq 3; g \in \langle B \rangle\}$. Since the last set of lines is closed under "multiplication" (i.e. under operations V) and a proper subset of L, Γ cannot generate L, contrary to $\langle \Gamma \rangle = L$. Consequently Γ should be infinite. □

Lemma 3.*Let* $\langle \Gamma \rangle = L$. *Then there is a* $\Delta \subset \Gamma$ *and* $\varphi : \Gamma \to \Delta$ *such that:*
(a) φ *is onto*, (b) $|\Delta \cap L_2| = 1$, (c) $|\Delta \cap L_3| = \emptyset$, (d) $\langle \Delta \rangle = L$

Proof. According to Lemma 2, at least one of $\Gamma \cap L_m$, $(m = 1,2,3)$ is infinite. At least two of them are nonempty or otherwise we cannot make a single "product". We will assume that $\Gamma \cap L_1$ is infinite, $(1,a) \in \Gamma \cap L_1$ and $(2,b) \in \Gamma \cap L_2$ for some $a, b \in G$.

For $(1,x) \in \Gamma$ let $\varphi(1,x) = (1,x)$ and $\varphi(2,b) = (2,b)$.

For $(2,y) \in \Gamma$ $(y \neq b)$ the lines $(1,a)$, $(2,y)$ and $(3,ay)$ are concurrent. The lines $(1,ayb^{-1})$, $(2,b)$ and $(3,ay)$ are also concurrent. So the lines $(2,y)$ and $(1,ayb^{-1})$ generate each other (using lines $(1,a)$ and $(2,b)$). Consequently we can replace $(2,y)$ by $(1,ayb^{-1})$ and still have a generating set. We define $\varphi(2,y) = (1,ayb^{-1})$.

For $(3,z) \in \Gamma$, the lines $(1,zb^{-1})$, $(2,b)$ and $(3,z)$ are concurrent. $(1,zb^{-1})$ and $(3,z)$ generate each other (using $(2,b)$), so $\varphi(3,z) = (1,zb^{-1})$.

If we define $\Delta = \varphi\Gamma$ the then (a)-(d) easily follow. $\square$

Lemma 4. *Let $\Gamma \cap L_1 = (1,X) = \{(1,x) | x \in X\}$, $\Gamma \cap L_2 = \{(2,b)\}$ and $\Gamma \cap L_3 = \emptyset$. Then $\langle\Gamma\rangle = L$ iff $\langle X\rangle = G$.*

Proof. (a) If $\langle\Gamma\rangle = L$ then $(1,G) \subset \langle\Gamma\rangle$. It is easy to prove by induction that all 1-lines generated by $(1,X)$ and $(2,b)$ belong to $(1, X \cdot \bigcup_{n=1}^{\infty}(X^{-1}X)^n)$. $S_0G \subset X \cdot \bigcup(X^{-1}X)^n \subset \langle X\rangle$.

(b) If $\langle X\rangle = G$ then the sets of lines generated by Γ are:
$(1, X \cdot \bigcup_{n=1}^{\infty}(X^{-1}X)^n) = (1, XG) = (1,G) = L_1$,
$(2, \bigcup_{n=1}^{\infty}(X^{-1}X)^n b) = (2, Gb) = (2,G) = L_2$,
$(3, X \cdot \bigcup_{n=1}^{\infty}(X^{-1}X)^n b) = (3, XGb) = (3,G) = L_3$.

Lemma 5. *Let $\langle\Gamma\rangle = L$. Tlaen there is a $\Delta \subset \Gamma$ such that $\Delta \neq \Gamma$ and $\langle\Delta\rangle = L$.*

Proof. We can assume $\Gamma \cap L_1 = (1,X)$, $\Gamma \cap L_2 = \{(2,b)\}$ and $\Gamma \cap L_3 = \emptyset$. Since $\langle\Gamma\rangle = L$ we get $\langle X\rangle = G$ by Lemma 4. But G has no minimal generating set so there is $Y \subset X$, $Y \neq X$ such that $\langle Y\rangle = G$. Then by Lemma 4 $\Delta = (1,Y) \cup \{(2,b)\}$ is a proper subset of Γ which generates L. $\square$

Theorem 2. *There is an infinite configuration without a base.*

Proof. Since every generating set of lines of the configuration M, defined above, has a proper subset which generates the same configuration, no generating set can be independent. Consequently there is no base for M.

We now give an example of an application of Belousov's algorithm where it does not produce a base of a configuration. Belousov's algorithm (algorithm $\mathcal{B}$ for short) chooses a set G of generators for the set of lines of a configuration M in the following manner.

To determine G we use a sequence $P_1, \ldots, P_m$ of points of M, such that

— the subsequence $P_1, \ldots, P_s$ has the property that every point P_r $(0 < r \leq s)$ is incident to a line l_r, not incident to any other point from ${}_1P, \ldots, P_s$.

— If $r > s$, the number of lines incident to P_r and some of the $P_1, \ldots, P_{r-1}$ is not less than the number of lines incident to P_q $(r < q \leq m)$ and some of $P_1, \ldots, P_{r-1}$.

Let G_s be a set of all lines of M incident to some of the points $P_1, \dots, P_s$. minus the set $\{l_1, \dots, l_s\}$. If G_{r-1} is defined $(s < r \leq m)$, then

— if only one line of the three lines incident to P_r is incident to some other point among $P_1, \dots, P_{r-1}, G_r$ is the union of G_{r-1} and one of the remaining two lines

— otherwise $G_r = G_{r-1}$,

then $G = G_m$, admissibly generates the lines of M.

Example 2. We will apply algorithm $\mathcal{B}$ to the following configuration:

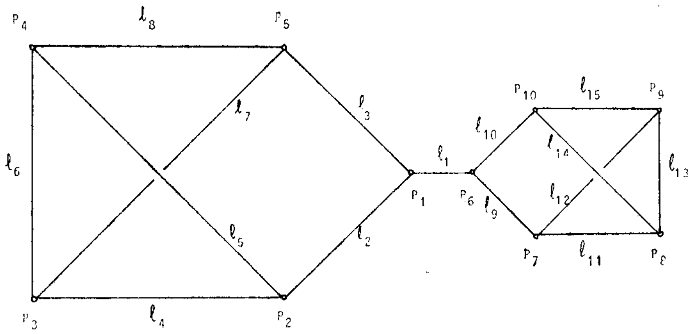

Fig. 1

Coordinates of points of the given configuration M are as follows: $P_1(6,2)$, $P_2(4,0)$, $P_3(0,0)$, $P_4(0,4)$, $P_5(4,4)$, $P_6(7,2)$, $P_8(10,1)$, $P_9(10,3)$ and $P_{10}(8,3)$. The set S of the coordinate system of the 4-web in which M is embedded, is $\{0, 1, \dots, 12\}$. Line classes consist of horizontal lines, vertical lines and lines $x + y = a$, $y = x + a$ $(a \in S)$ where $+$ is addition mod 13.

Applying algorithm $\mathcal{B}$ we get $G_1 = \{l_1, l_2\}$, $G_2 = \{l_1, l_2, l_4\}$ etc. Finally $G = G_{10} = \{l_1, l_2, l_4, l_6, l_9, l_{11}, l_{13}\}$. But G is not independent since l_1 can be generated by either $\{l_2, l_4, l_6\}$ or $\{l_9, l_{11}, l_{13}\}$.

The following theorem shows why Belousov's algorithm does not produce a base in the above example. Notice that configuration M of Example 2 is (3, 2)-homogeneous.

A line l of a configuration M is a singular line iff:

— it is incident to exactly two points P and Q

— both $M \setminus \{P\}$ and $M\{Q\}$ are not connected.

THEOREM 3. *No base of a* $(3,2)$*-homogeneous configuration contains a singular line.*

Proof. Let l be a singular line in a $(3,2)$-homogeneous configuration M, and let P_1, P_2 be the two points on l.

Assume there exists a base B of M containing l. Then $B \setminus \{l\}$ can be decomposed into two disjoint nonempty subsets B_1 and B_2, where B_1 is the set of all base lines incident to points connected to P_1, when P_2 is deleted from the points of M, and B_2 is similarly defined by interchanging the subscripts.

Consider now the two lines other than l which are incident to P_2. At least one of these must be generated by base lines from $B \setminus \{l\}$ and at the same time the generating set cannot contain a line from B_1 because l is a singular line. Consequently at least one of the lines through P_2 is generated by B_2. If both lines through P_2 (other than l) are generated by B_2 then we have a contradiction i.e. l is generated by B_2. Suppose one of the two lines is not generated by B_2 and let P_3 be the second point incident to that line. Again we find that of the two other lines incident to P_3 at least one must be generated by B_2 and if both are, again we have a contradiction. Continuing the process we can find a sequence of distinct points $P_2, \dots, P_i$; such that each $P_2, \dots, P_{i-1}$ is incident to exactly one new line generated by B_2. As M has a finite number of points, the process must terminate and there will occur a point P_i which is incident to two lines generated by B_2. This leads to the contradiction that all the lines incident to P_{i-1} are generated by B_2.$\square$

We now define another algorithm for removing dependent lines from a generating set of lines of a configuration. We call it the algorithm $\mathcal{H}$.

Let P be a point of a configuration M and $(i_1, a_1), \dots, (i_m, a_m)$ all lines of M incident to P. If $i_1 < i_2 < \dots < i_m$, then $\Sigma(P) = \{V_{i_1 i_2 i_j}(a_1, a_2) = a_j | \, 3 \le j \le m\}$ and $\Sigma_0 = \bigcup_{P \in M} \Sigma(P)$.

Let Σ_n $(n \in N)$ be given so that there is at least one equality E_n in it with a constant a occurring only once in E_n. Then we can solve the equation E_n in a. If this solution is $a = t_n$, we get Σ_{n+1} by replacing all occurrences of a in $\Sigma_n \setminus \{E_n\}$ by t_n. Obviously all terms in Σ_{n+1}, are admissible.

If in Σ_n there is no constant occurring in some equality only once, then $\Sigma = \Sigma_n$.

Let Γ be the set of all lines (i, a) such that the constant a admissible for (i, a) occurs in Σ.

THEOREM 4. *If L is the set of' all lines of a configuration M and Γ a set of lines of M produced by algorithm $\mathcal{H}$ then $\langle \Gamma \rangle = L$.*

Proof. Trivial.

Before we attack the real problem as to whether the algorithm $\mathcal{H}$ produces a base of a configuration, we give some applications.

THEOREM 5. *The algorithm $\mathcal{H}$ produces a base for any finite $(3,2)$-homogeneous configuration of a web W.*

We will need the following Lemma for the proof of Theorem 5.

LEMMA 6. (Belousov) *If M is a finite configuration of a web W with $\pi = |M|$ points and $\lambda = |L|$ lines, Γ a generating set of lines for M with γ lines and ε the number of nontrivial equalities among lines of Γ then $\gamma = \varepsilon + \lambda - \pi$.*

Proof of the Theorem 5: Since there are exactly three lines of M incident to every point of M and exactly two points of M incident to every line of a $(3,2)$-homogeneous configuration M, there is a number α such that $\lambda = 3\alpha$ and $\pi = 2\alpha$.

We now apply algorithm $\mathcal{H}$ to M. In Σ_0 there are 2α equations (one for every point of M) with 3α constants (one for every line of M) each of them occurring exactly twice in Σ_0. In every step of the algorithm $\mathcal{B}$ we loose one equation and one constant. By removing the equation E_n from Σ_n we also remove some constants. All except a are reintroduced to $\Sigma_n \setminus \{E_n\}$ when we replace a in $\Sigma_n \setminus \{E_n\}$. So every constant from Σ_{n+1} occurs exactly twice in it.

Since two points P, Q of M are connected in M iff there is a sequence of equations, the first corresponding to P, the last to Q, such that two consecutive equations have a constant in common (collinearity) and since every configuration is connected, we can apply algorithm $\mathcal{H}$ until only one equation remains.

For otherwise there would be more than one equation, each with the property that every constant in it occurs exactly twice, so they have no constants in common, contrary to connectedness.

So $\Sigma_{2\alpha-1}$ is one equation with $\alpha+1$ constants, each of them occurring exactly twice in $\Sigma_{2\alpha-1}$. Let Γ be the set of corresponding lines defined by algorithm $\mathcal{H}$ and $B \subset \Gamma$ a base. Then $|B| \leq |\Gamma| = \alpha + 1$. Since there is at least one equation among base lines, $\varepsilon \geq 1$ so by Lemma 6 $|B| = \varepsilon + \lambda - \pi = \varepsilon + \alpha \geq 1 + \alpha$. Consequently $|B| = \alpha + 1$ and $B = \Gamma$ so algorithm $\mathcal{H}$ actually produces a base for M. □

Corolary 2. *All bases of a given $(3,2)$-homogeneous configuration contain the same number of lines.*

Problem 1. Does Corollary 2 hold if the condition of $(3,2)$-homogeneity is removed?

Following [bf 5] we define a strictly quadratic equation as one in which every variable appears exactly twice. So we have:

Corollary 3. *If a $(3,2)$-homogeneous configuration M can be embedded everywhere in a web W then a strictly quadratic equation holds in tire covering algebra of W.*

Proof. If M can be embedded everywhere in W this means that the constants occurring in $\Sigma = \Sigma_{2\alpha-1}$ can be replaced by variables. The proof of Theorem 5 shows that Σ is a strictly quadratic equation in the covering algebra of W. □

The converse of Corollary 3 is also true for "reasonable" strictly quadratic equations. The proof goes beyond the scope of the present article (we need a notion of a free web among others) and will be published elsewhere.

Even though the algorithm $\mathcal{H}$ produces a base for any $(3,2)$-homogeneous configuration, this is not true in general. The following example shows that algorithm $\mathcal{H}$ fails to produce a base in some cases.

Example 3. We will apply algorithm $\mathcal{H}$ to the following configuration:

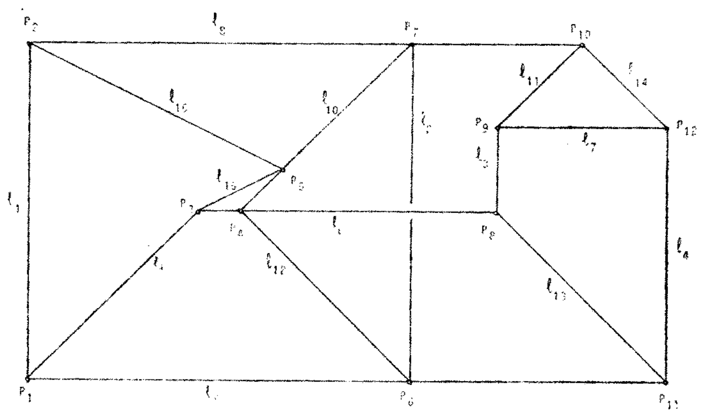

Fig. 2

Coordinates of points of the given configuration M are as follows: $P_1(0,0)$, $P_2(0,8)$, $P_3(4,4)$, $P_4(5,4)$, $P_5(6,5)$, $P_6(9,0)$, $P_7(9,8)$, $P_8(11,4)$, $P_9(11,6)$, P_{10} $(13,8)$, $P_{11}(15,0)$ and $P_{12}(15,6)$. The set S of the coordinate system of the 6-web in which M is embedded, is $\{0,1,\ldots,16\}$. Line classes consist of vertical lines, horizontal line sand lines $y = x + a$, $x + y = a$, $y = 9x + a$, $y = 8x + a$ $(a \in S)$ where $+$ is addition mod 17.

Applying algorithm $\mathcal{H}$ we get:
$\Sigma_0 = \{V_{123}(a_1,a_5) = a_9,\ V_{126}(a_1,a_8) = a_{16},\ V_{235}(a_6,a_9) = a_{15},\ V_{234}(a_6,10) = a_{12},$ $V_{356}(a_{10},a_{15}) = a_{16},\ V_{124}(a_2,a_5) = a_{12},\ V_{123}(a_2,a_8) = a_{10},\ V_{124}(a_3,a_6) = a_{13},$ $V_{123}(a_3,a_7) = a_{11},\ V_{234}(a_8,a_{14}) = a_{14},\ V_{124}(a_4,a_5) = a_{13},\ V_{124}(a_4,a_7) = a_{14}$ where a_m is the constant for the second coordinate of the line l_m $(1 \le m \le 16)$.

Since $a_8 = V_{123}(a_2,a_{10})$ we get $\Sigma_1 = \{V_{123}(a_1,a_5) = a_9,\ V_{162}(a_1,a_{16}) = V_{132}(a_2,a_{10}),\ V_{235}(a6,a_9) = a_{15},\ V_{234}(a_6,a_{10}) = a_{12},\ V_{356}(a_{10},a_{15}) = a_{16},$ $V_{124}(a_2,a_5) = a_{12},\ V_{124}(a_3,a_6) = a_{13},\ V_{123}(a_3,a_7) = a_{11},\ V_{132}(a_2,a_{10}) = V_{342}(a_{11},a_{14}),\ V_{124}(a_4,a_5) = a_{13},\ V_{124}(a_4,a_7) = a_{14}\}$. Next we solve the appropriate equations for $a_1, a_{16}, a_2, a_{11}, a_{14}, a_{15}, a_{12}, a_3$ and a_4, finally to get

$\Sigma_{10} = \{V_{152}(V_{231}(a_5,a_9),\ V_{356}(a_{10},V_{235}(a_6,a_9))) = V_{132}(V_{241}(a_5,\ V_{234}(a_6,a_{10})),$ $a_{10}),\ V_{132}(V_{241}(a_5,\ V_{234}(a_6,a_{10})),a_{10}) = V_{342}(V_{123}(V_{241}(a_6,a_{13}),a_7),\ V_{124}(V_{241}(a_5,$ $a_{13}),a_7))\}$

No constant occurs in either equation only once so the algorithm $\mathcal{H}$ cannot be applied further. However the equation

$$V_{l62}(V_{231}(a_5,a_9),\ V_{356}(a_{10},\ V_{235}(a_6,a_9))) = V_{342}(V_{123}(V_{241}(a_6,a_{13}),a_7)$$
$$V_{124}(V_{241}(a_5,a_{13}),a_7))$$

is a simple consequence of Σ in which a_{10} appears only once. Consequetly it can be solved for a_{10} showing that the set of lines $\Gamma = \{l_5, l_6, l_7 l_9, l_{10}, l_{13}\}$ obtained by algorithm $\mathcal{H}$ is not independent. $\square$

Therefore we would like to pose the following:

Problem 2. Find an efficient algorithm which will produce a base for a given finite configuration of a web.

Although the word "efficient" is essential here, we cannot define it precisely but we hope it will be clear what we mean by it from the following:

— we consider the algorithm $\mathcal{H}$ as being efficient in the case of $(3,2)$-homogeneous configurations

— let us define algorithms $\mathcal{B}'$ and $\mathcal{H}'$ by adding the following step to the algorithms $\mathcal{B}$ and $\mathcal{H}$ respectively: remove all dependent lines from the set of generating lines obtained by algorithm $\mathcal{B}(\mathcal{H})$. We consider both algorithms $\mathcal{B}'$ and $\mathcal{H}'$ inefficient (even if they do produce a base of a given configuration) since it is necessary to check for independence all (or at least many) subsets of a given set of lines.

Finally we would like to thank S. Krstić for his help related to Example 1.

This work was supported in part by Canadian N. S. E. R. C. grant A8144.

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(Received 03 10 1984)
(Revised 24 OS 1985)