PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 38 (52), 1985, pp. 45–49

ON SOME RADICALS IN NEAR-RINGS WITH A DEFECT OF DISTRIBUTIVITY

Vučić Dašić

Abstract. We consider some properties of the radical $J_2(R)$ and the Levitzki radical $L(R)$ in a near-ring R with a defect of distributivity. With and additional assumption that the defect D of R is nilpotent or D is contained in the commutator subgroup of $(R, +)$ we generalize some results of Freidman [**6**, Theorems 1, 2], and of Beidleman [**1**, Th. 16]. Also, we give a slight version of the Theorem 2.5 of [**3**]. By using the notation of a relative defect, we consider some properties of minimal nonnilpotent R-subgroups and we generalize some results of Beidleman [**2**, Theorems 2.4, 2.6, 2.7, 3.1].

1. Preliminaries

A left zero-symmetric near-ring R is a set with two binary operations $+$ and $\cdot$ such that

(1°) $(R, +)$ is a group (not necessarily abelian)

(2°) $(R, \cdot)$ is a semigroup

(3°) The left distributivity law holds, i.e.

$$x(y + z) = xy + xz, \quad \text{for all } x, y, z \in R.$$

Also we suppose $0x = 0$ for all $x \in R$.

Let R be a near-ring and let $(S, \cdot)$ be a multiplicative subsemigroup of $(R, \cdot)$ whose elements generate $(R, +)$. We say that S is a set of generators of the near-ring R. Thus, every element $r \in R$ can be represented as a finite sum $\sum_i(\pm s_i)$, $(s_i \in S)$. Denote by $D = D(S)$ the normal subgroup of the group $(R, +)$ generated by the set $\{d : d = -(xs + yx) + (x + y)s,\ x, y \in R, s \in S\}$. It was proved in [**4**] that D is an ideal of R. If $S \subset R$ is a proper subset of R, then we say that R is a near-ring with the defect of distributivity D. If we wish to stress the set of generators, then we write (R, S). Thus, in the near-ring (R, S) with the defect D, for all $x, y \in R$ and $s \in S$ there exists $d \in D$ such that $(x + y)s = xs + ys + d$. Specially, if $D = \{0\}$ then R is a distributively generated (briefly d.g.) near-ring. If

---

AMS Subject Classification (1980): Primary 16A76.

$S = R$, then we say that R is a D-distributive near-ring and then for all $x, y, z \in R$ there exists $d \in D$ such that $(x + y)z = xz + yz + d$. Specially, if $D = \{0\}$ then R is a distributive near-ring.

A right ideal K of R is a normal subgroup of $(R, +)$ such that $(x+a)y-xy \in K$ for all $a \in K$, $x, y \in R$. An ideal H of R such that $ra \in H$ for all $a \in H$, $r \in R$. An R-subgroup B of R is subgroup $(R, +)$ such that $br \in B$ for all $b \in B$, $r \in R$.

Let A be a nonempty subset of R and let A' be the normal subgroup of $(R, +)$ generated by A. The normal subgroup $D_r(A)$ of $(R, +)$ generated by the elements of the form

$$d = -(xs + a's) + (x + a')s, \quad (x \in R, a' \in A', s \in S)$$

is called the relative defect of the subset A with respect to R. In [**5**] it was proved that the relative defect $D_r(A)$ of some ideal A of R is an ideal of R too.

2. Some properties of the radical $J_2(R)$ and the Levitzki radical

A right ideal B of a near-ring R is called modular (strictly maximal) if B is maximal as an R-subgroup. Let I denote the colletion of all modular right ideals of R. We define the radical $J_2(R)$ of R by radical $J_2(R)$ of R by $J_2(R) = \bigcap_{B \in I} B$. The radical subgroup $R_s(R)$ of R is the intersection of all maximal R-subgroup of R.

In a zeror-symmetric near-ring R every right ideal is an R-subgroup, hence a subnear-ring. We recall that a near-ring R is locally nilpotent if for every finite subset H of R there exists a positive integer $n = n(H)$ such that the product of every n elements from H is zero. An ideal of R is locally nilpotent if it is locally nilpotent as a near-ring ideals of R is called the Levitzki radical $L(R)$ of R.

The following two results generalize respectively theorems 1 and 2 of [**6**]. Namely, we extend these results of Friedman about distributive near-rings to a wider class of D-distributive near-rings. In addition we only require that the defect D is nilpotent (Th. 2.1) and that the defect D is contained in the commutator subgroup of R (Th. 2.2).

THEOREM 2.1. *Let A be an ideal of a D-distributive near-ring R with a nilpotent defect D: Then the Levitzki radical $L(A)$ of A is a locally nilpotent ideal of R and $L(A) = L(R) \cap A$.*

Proof. The relative defect $D_r(A)$ of an ideal A is an ideal of R too, and it is contained in the defect D [**5**, Th. 4]. Thus, $D_r(A)$ is nilpotent. Hence, $D_r(A) \subseteq L(A)$, i.e. $D(A) \subseteq D_r(A) \subseteq L(A)$, where $D(A)$ is the defect of the ideal A, considering A as a near-ring. By Proposition 5 of [**5**], $A/L(A)$ is a distributive near-ring and has no non-zero locally nilpotent ideals. Using Lemma 3 in [**6**], it follows that $L(A)$ is a locally nilpotent ideal of R. Thus, $L(A) \subseteq L(R) \cap A$. Also, $L(R) \cap A$ is a locally nilpotent ideal of A. Consequently $L(R) \cap A \subseteq L(A)$ i.e. $L(R) \cap A = L(A)$.

THEOREM 2.2. *Let R be a D-dislributive near-ring with a nilpotent defect D which is contained in the commutator subgroup R' of $(R, +)$. Then the factor near-ring $R/L(R)$ is a ring, where $L(R)$ is the Levitzki radical of R.*

Proof. Since D is a nilpotent ideal of R we have $D \subseteq L(R)$. Thus, by Proposition 5 ot [**5**], $R/L(R)$ is distributive. On the other hand, by Theorem 2 of [**5**], R' is a nilpotent ideal of R, i.e. $R' \subseteq L(R)$. Therefore, $R/L(R)$ is an abelian group with respect to addition; thus $R/L(R)$ is a ring.

An ideal B of a near-ring R is called strictly small if and only if $R = C$ for each R-subgroup C such that $R = B + C$. The following result generalizes Theorem 16 of [**1**]. Namely, we extend this result of Beidleman about d.g. near-rings to a wider class of near-rings with a defect of distributivity. In this goal we only require that the defect D is contained in the commutator subgroup of R.

THEOREM 2.3. *Let R be a near-ring whose defect D is contained in the commutator subgroup R' of $(R,+)$. If $(R,+)$ is a finitely generated nilpotent group, then the radical $J_2(R)$ is a strictly small ideal.*

Proof. In view of Theorem 6 [**1**], we need to show that $J_2(R) = \bigcap_{B\in I'} B$, where I' is the set of maximal R-subgroups. Thus it suffices to show that every maximal R-subgroup of R is right ideal of R too. Let B be a maximal R-subgroup of R. Since $(R,+)$ is finitely generated, it follows that there exists a maximal sugroup B_1 of $(R,+)$, $B \subseteq_1$B,. By Corollary 10. 3.2. of [**7**] it follows that B_1 is a proper normal subgroup of $(R,+)$ and $R' \subseteq B$,and so $D \subseteq B_1$. If B_2 the normal subgroup of $(R,+)$ generated by the set B, then it is easy to show that $B_2 + D$ is a right ideal of R and $B \subseteq B_2 + D \neq R$. Since B is maximal R-subgroup of R, it follows that $B = B_2 + D$, whence B is a right ideal of R.

We now give a slightly modified version of some earlier results [**3**, Corollary to Th. 2.5]. We say that B is a small normal subgroup of $(R,+)$ if and only if $R = C$ for each normal subgroup C of $(R,+)$ such that $(R,+) = (B,+) + (C,+)$.

THEOREM 2.4. *Let R be a near-ring whose defect D is a small normal subgroup of $(R,+)$. If $(R,+)$ is a nilpotent group and R has the identity, then $J_2(R) = R_s(R)$.*

Proof. Obviously $R_s(R) \subseteq J_2(R)$. We need to show that $J_2(R) \subseteq R_s(R)$. Let B be a maximal R-subgroup. It is well known [**9**, Th. 6.4.10.] that B is a term of a normal series for $(R,+)$. Thus, B is contained in a proper normal subgroup C of $(R,+)$. But, $C+D$ is a normal subgroup of $(R,+)$ and $C+D \neq R$, because $C \neq R$ and D is a small normal subgroup of $(R,+)$. Thus, there exists a proper normal subgroup $C + D$ of $(R,+)$ containing B and D. Therefore, the normal subgroup B_1, of $(R,+)$ generated by the set B is contained in $C + D$, so $B_1 + D \neq R$. It is easy to see that $B_1 + D$ is a right ideal of R which contains the R-subgroup B. Since B is a maximal R-subgroup, it follows that $B = B_1 + D$, i.e. B is a right ideal of R. Thus, every R-subgroup is a right ideal of R.

COROLLARY. *Let R be a near-ring whose defect D is a small normal subgroup of $(R,+)$. If $(R,+)$ is a nilpotent group and R has the identity, then the radical $J_2(R)$ is a quasi-regular ideal of R.*

Throughout this section we shall assume that R satisfies the descending chain condition on R-subgroups, R has the identity and the radical $J_2(R)$ is a nilpotent ideal.

An R-subgroup B of a near-ring R is called minimal nonnilpotent if B is nonnilpotent and every proper R-subgroup in B is nilpotent. A proper right ideal B of R is said to be complemented in R if there exists an R-subgroup H of R such that $R = B + H$ and $B \cap H = \{0\}$.

We first need the following

PROPOSITION 3.1. *Let B be an R-subgroup and let A be a right ideal of R. If the relative defect of the subset B is contained in B, then $B \cap A$ is a right ideal of R.*

Proof. We have only to show that the relative defect of the subset $b \cap A$ is contained in $B \cap A$. By definition of the relative defect, $D_r(B \cap A)$ is generated by all elements d in R for which there exist $x \in R$, $s \in S$ and $b \in B \cap A$ such that $d = -bs - xs + (x + b)s$. Since $D_r(B) \subseteq B$, it follows that $d \in B$. By Lemma 2.3. of [**4**], we have $d \in A$, because A is a right ideal of R. Thus, for all $d \in D_r(B \cap A)$, it follows that $d \in B \cap A$, i.e. $D_r(B \cap A) \subseteq B \cap A$. Therefore by Lemma 3.2. of [**4**], we have that $B \cap A$ is a right ideal of R.

The following results are generalizations of some results of Beidleman [**2**, Theorems 2.4, 2.6, 3.1]. The results of Beidleman refer to a class of d.g. near-rings. We transmit these results over a wider class of near-rings with a defect of distributivity. Here we only impose an additional condition of the form: every minimal nonnilpotent R-subgroup contains the relative defect of its own (Theorems 3.2, 3.3.), or every nonnilpotent R-subgroup contains the relative defect of its own (Theorems 3.4, 3.5).

THEOREM 3.2. *Let R be a near-ring with a defect of distributivity and let every minimal nonnilpotent R-subgroup of a near-ring R contain the relative defect of its own. If B is a minimal nonnilpotent R-subgroup of R, then $B \cap J_2(R)$ is the unique strictly maximal right ideal of B.*

Proof. From Proposition 3.1. it follows that $B \cap J_2(R)$ is an ideal of B. The proof of the remaining part is the same as that of Theorem 2.4. of [**2**].

THEOREM 3.3. *Let R be a near-ring with a defect of distributivity and let every minimal nonnilpotent R-subgroup of R contain the relative defect of its own. Further, let $f : R \to R_2$ denote the natural near-ring homomorphism of the near-ring R onto the near-ring $R_2 = R/J_2(R)$. If B is a minimal nonnilpotent R-subgroup of R, then $(B)f$ is a minimal R_2 subgroup of R_2.*

Proof. By the First isomorphism theorem we have

$$(B + J_2(R))/J_2(R) \simeq B/B \cap J_2(R).$$

On the other hand $(B)f = (B + J_2(R))f = (B + J_2(R))/J_2(R)$.

From Theorem 3.2. it follows that $B \cap J_2(R)$ is a strictly maximal ideal of B. Thus, $(B)f$ is a minimal R-subgroup of the R-group R_2, i.e. $(B)f$ is a minimal R_2-subgroup of R_2.

THEOREM 3.4. *Let R be a near-ring with a defect of distributivity and let every nonnilpotent R-subgroup of R contains the relative defect of its own. A nonnilpotent R-subgroup B of R is minimal nonnilpotent if and only if B contains no proper nonzero normal R-subgroups which are complemented in B.*

Proof. Assume that B is a minimal nonnilpotent R-subgrup. Let B_1 be a proper nonzero normal R-subgroup of B that is complemented in B by an R-subgroup $B_2 \subseteq B$. From definition of a minimal nonnilpotent R-subgroup, it follows that the R-subgroups B_1 and B_2 are nilpotent. The radical $J_2(R)$ contains all nilpotent R-subgroups [**8**, Corollary 5.45]. Since $J_2(R)$ nilpotent, we have that $B = B_1 + B_2$ is nilpotent whih contradicts the assumption above. Conversely, let a nonnilpotent R-subgroup B contains no proper nonzero normal R-subgroups that are complemented in B. We assume that there exists a minimal nonnilpotent R-subgroup C which is contained in B and we seek a contradiction to this assumption. Namely, by using Theorem 3.51 of [**8**], C contains an idempotent c such that $cR = C$ and $R = cR + A(c)$, $A(c) = \{r : cr = 0, r \in R\}$. Hence, $B = cR + A(c) \cap B$. Since $A(c)$ is a right ideal of R, it follows that $A(c) \cap B$ is a normal R-subgroup of the R-group B. From Proposition 3.1. we have that $A(c) \cap B$ is a right ideal of R, i.e. $A(c) \cap B$ is a right ideal of B. The proof of the remaining part is the same as that of the Theorem 2.7 in [**2**].

THEOREM 3.5. *Let R be a near-ring with a defect of distributivity and let every nonnilpotent R-subgroup of R contains the relative defect of its own. Further, let $R_2 = R/J_2(R)$ be a ring and let b be an idempotent of R. Then $B = bR = cR + A(c) \cap B$, where c is an idempotent element of R contained in B. Moreover, the R-subgroup $A(c) \cap B$ is nilpotent if and only if, $b_2R_2b_2$ is a division ring, where $b_2 = (b)f$ (f is the natural near-ring homomorphism of R onto R_2).*

Proof. The proof is the same as that of Theorem 3 of [**2**], whereby we use as jet the result of the Proposition 3.1 and the result of the Theorem 3.4.

The author is indebted to Professor V. Perić for some helpful comments.

REFERENCES

[1] J.C. Beidleman, *A radical for near-ring modules*, Michigan Math. J. **12**(1965), 377–383.

[2] J.C. Beidleman, *Nonsemisimple distributively generated near-rings with minimum condition*, Math. Annalen **170**(1967), 206–213.

[3] V. Dašić, *On the radicals of near-rings with defect of distributivity*, Publ. Inst. Math. (Beograd) **28(43)**(1980), 51–59.

[4] V. Dašić, *A defect of distributivity of the near-ring*, Math. Balkanica **8: 8**(1978), 63–75.

[5] V. Dašić, *Some properties of the defect of distributivity of a near-rings*, Third Algebraic Conference Beograd, 1982, 67–71.

[6] P.A. Freidman, *Distributively solvable near-rings*, Proc. Riga Seminar on Algebra (Russian), Riga, 1969, 279–309.

[7] M. Hall, *The Theory of Groups*, Macmillan, New York, 1969.

[8] G.G. Pilz, *Near-rings*, Horth-Holland, Amsterdam, 1977.

[9] W.R. Scott, *Group Theory*, Prentice Hall, New Jersey, 1964.

Institut za matematiku i fiziku
Cetinjski put bb
81000 Titograd
Yugoslavia

(Received 16 03 1984)
(Revised 07 03 1985)