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ON A THEOREM OF ŠUTOV

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Abstract. This note deals with formulas occurring in Mal'cev's and Šutov's axiomatizations of the class of semigroups embeddable in a group. Assuming α and β are schemes as defined by Mal'cev and $T(\alpha)$, $T(\beta)$ corresponding Mal'cev quasi-identities and $T(\beta, x)$ the Šutov quasi-identity arising from $T(\beta)$ it is proved that there exists a semigroup on which $T(\beta, x)$ is true and $T(\alpha)$ is not whenever α is irreducible and $|\alpha| > |\beta|/2 + 2$.

In a couple of famous articles [**1**] and [**2**] Mal'cev considered the problem of embedding a semigroup in a group, found an infinite axiomatization of the class of embeddable semigroups, and proved that this class is not finitely axiomatizable. A similar problem of potential invertibility of elements of semigroups i.e., given a semigroup S and an element a of S, whether S can be embedded in a semigroup in which a has an inverse) was considered by Šutov [**3**].

Finite sequences of elements ± 1, ± 2 which satisfy certain properties are called schemes in [**1**] and for every scheme a an associated quasi-identity $T(\alpha)$ is defined [**1**, p. 336]. These $T(\alpha)$'s constitute the Mal'cev's aximatization. $T(\alpha)$ is written in variables $a_i, b_i, l_i, r_i, A_j, B_j, L_j, R_j$ where i $i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$ and p and q are respectlvely the numbers of occurrences of 1 and 2 in α. Šutov [**3**] defines $T(\alpha, x)$ to be the quasi-identity obtained from $T(\alpha)$ by replacing all variables l_i, L_i by a new variable x. He showed that an element a of a semigroup S is potentially invertible iff S satisfies all quasi-identities $T(\alpha, a)$. In a subsequent paper [**4**] Šutov proved that a semigroup is embeddable in a group iff every its element is potentially invertible. As a consequence, the set of all quasi-identities $T(\alpha, x)$ axiomatizes the class of embeddable semigroups. A natural question now is whether this new axiomatization is a substantial refinement of the old one, i.e. whether $T(\alpha, x)$ is a strict consequence of $T(\alpha)$. Šutov proved that in many cases it is indeed so and we quote

THEOREM 8. of [**4**]. *If* $\alpha| \leq 44$ *then* $T(\alpha, x) \vdash T(\alpha)$. *For every* $n > 2$ *there exists a scheme* α_n *with* $|\alpha_n| = 2n$ *and* $T(\alpha_n, x) \vdash T(\alpha)$.

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Here $|\alpha|$ denotes the length of α and $\Sigma \vdash \Theta$ stands for "Θ is true on every semigroup on which Σ is true".

The aim of this note is to prove the following more general

THEOREM. *If α und β are schemes and α is irredncible then $T(\beta,x) \vdash T(\alpha)$ implies $|\alpha| \leq |beta|/2+2$.*

Irreducibility here is as in [**2**]: a scheme is irreducible if no proper segment of it is itself a scheme. Since it is easy to construct an irreducible scheme of any (even) length the non-trivial part of Šutov's theorem follows from the special case $\alpha = \beta$ of our theorem: if α is irreducible and $|\alpha| > 4$ then $T(\alpha,x)HT(\alpha)$.

We remark that the following proof depends on Lemmas 2 and 3 of [**2**] and is much shorter than the proof of theorem 8 in [**4**]. The reader is supposed to have some familiarity with [bf 1] and [**2**]. We shall make free use of Mal'cev's terminology and will not bother to repeat the definitions. However, a precise reference will be given for every unexplained notion.

Proof. Suppose $T(\beta,x) \vdash T(\alpha)$, $T(\beta,x) = (\sigma_1 \wedge \cdots \wedge \sigma_m \Rightarrow \sigma_0)$, and $T(\alpha) = (\psi_1 \wedge \cdots \wedge \psi_n \Rightarrow \psi_0)$. Then $|\alpha| = m+1$ and $|\beta| = n+1$. Let S_α be the semigroup the generating set of which is the set X of all variables involved in $T(\alpha)$ and the set of defining relations is $\{\psi_1,\ldots,\psi_n\}$. Then $T(\alpha)$ is not true on S_α [**2**, p. 259) and so $T(\beta,x)$ is not true on S_α either.

For every word u over X there exists a word u_0 in normal form [**2**, p. 259] such that $u = u_0$ in S_α. Therefore, there is a mapping φ which assigns to every variable occurring in $T(\beta,x)$ a word in normal form over X and such that $\sigma_i^\varphi,\ldots,sigma_m^\varphi$ are true in S_α and σ_0^φ is not. (Here σ_i^φ denotes σ_i with all variables replaced by their φ-values.)

Now every identity σ_i^φ is of the form $u_1v_1 = u_2v_2$ with u_j, v_j in normal form. We say that σ_i^φ is trivial if u_1u_1 and v_2v_2 are the same words. If σ_i^φ is not trivial and $i > 0$ then it is easy to see that $u_1 = ux_1$, $u_2 = ux_2$, $v_1 = u_1v$, $v_2 = y_2v$ for some $x_1,x_2,y_1,y_2 \in X$ and some (possibly empty) words, u,v over X. Moreover, the identity $\sigma_i = (x_1y_1 = x_2y_2)$ is one of $\psi_1,\ldots,\psi_n$. Without a loss of generality we may assume σ_i is not trivial for $1 \leq i \leq m' \leq m$ and is trivial for all other (if any) values of i. Thus, $\{\overline{\sigma}_1,\ldots,\overline{\sigma}_{m'}\} \subseteq \{\psi_1,\ldots,\psi_n\}$.

Since σ_0^φ is a group consequence [**2**, p. 253] of the set $\{\sigma_1^\varphi,\ldots,\sigma_m^\varphi\}$ (as σ_0 is a group consequence of $\{\sigma_1,\ldots,\sigma_m\}$) it follows that σ_0^φ is a group consequence of $\{\overline{\sigma}_1,\ldots,\overline{\sigma}_{m'}\}$. From Lemmas 2 and 3 of [**2**] it follows that all group consequences of a proper subset of $\{\psi_1,\ldots,\psi_n\}$ are simple consequences [**2**, p. 253]. Not being true in Sa the identity σ_0^φ is not a simple consequence of $\{\psi_1,\ldots,\psi_n\}$ and so we get $\{\overline{\sigma}_1,\ldots,\overline{\sigma}_{m'}\} \subseteq \{\psi_1,\ldots,\psi_n\}$.

Let p and q be respectively the numbers of occurrences of 1 and 2 in β. Suppose $p \leq q$; the other case will follow by symmetry. Let $T(\beta) = (\sigma_1' \wedge \cdots \wedge \sigma_m' \Rightarrow \sigma_0')$ so that σ_i is obtained from σ_i' by replacing every l_j and L_j by x. From the table defining $T(\beta)$ [**1**, p. 336] it follows that if L_j occurs in σ_i then it occurs there only as a left factor and that no L_j occurs in σ_0'. Moreover, since 2, -2 do not occur in this order as adjacent element in β it follows that every σ_i' contains at most one occurrence of at most one L_j. Thus, since q is the number of L-variables occurring in $T(\beta)$ there are exactly $2q$ identities among $\sigma_1', \dots, \sigma_n'$ which contain an occurrence of one L-variable. Hence if ξ is the last letter of the word $\varphi(x)$ then ξ occurs as a left factor in exactly $2q$ identities among $\sigma_1, \dots, \sigma_m$. Consequently ξ occurs as a left factor in at least $2q - (m - m')$ identities among $\{\overline{\sigma}_1, \dots, \overline{\sigma}_{m'}\}$. Since every variable occurs at most twice in $\{\psi_1, \dots, \psi_n\}$ it follows that among $\overline{\sigma}_1, \dots, \overline{\sigma}_{m'}$ there are at least $2q + m' - m - 2$ redundant (in the sense of repetition) identities. Recalling that every ψ_i $(1 \leq i \leq n)$ is an element of $\{\overline{\sigma}_1, \dots, \overline{\sigma}_{m'}\}$ it follows that $m' - (2q + m' - m - 2) \geq n$ and so $n \leq m - 2q + 2$. Now $2q \geq p + q = |\beta|/2$ and the desired inequality $|\alpha| \leq |\beta|/2 + 2$ immediately follows.

REFERENCES

[1] A.I. Mal'cev, *On the immersion of associative systems in groups* (Russian), Mat. Sbornik (N. S) **6**(1939), 331–336.

[2] A.I. Mal'cev, *On the immersion of associative systems in groups II* (Russian), ibid. **8**(1940), 251–264.)

[3] E.G. Šutov, *On potential invertibility of elements of semigroups* (Russian), Izv. vuzov, Matematika **53**(1966), 153–163.

[4] E.G. Šutov, *Emtedding semigroups in groups and potential invertibility* (Russian), Sibir. Mat. Ž. **21**(1980), 168–180.

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