AN APPLICATION OF THE RUSCHEWEYH DERIVATIVES II

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Abstract. We introduce the class \( R(\alpha) \) of functions of the form

\[
f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0)
\]

satisfying the condition

\[
\Re\{D^{n+1}f(z)/D^n f(z)\} > \alpha/(\alpha + 1)
\]

for some \( \alpha \geq 0 \) and for all \( z \in U = \{z : |z| < 1\} \), where \( D^n f(z) \) denotes the Hadamard product of \( z/(1-z)^{n+1} \) and \( f(z) \). The object of the present paper is to prove some distortion and closure theorems for functions \( f(z) \) in \( R(\alpha) \), and to give the result for the modified Hadamard product of functions \( f(z) \) belonging to the class \( R(\alpha) \). Furthermore, we determine the radii of starlikeness and convexity of functions \( f(z) \) in the class \( R(\alpha) \).

1. Introduction

Let \( A \) denote the class of functions \( f(z) \) of the form

\[
f(z) = \sum a_k z^k
\]

which are analytic in the unit disk \( U = \{z : |z| < 1\} \). We denote by \( S \) the subclass of univalent functions \( f(z) \) in \( A \), and by \( S^* \) and \( K \) the subclasses of \( S \) whose members are starlike with respect to the origin and convex in the unit disk \( U \), respectively. A function \( f(z) \) belonging to the class \( A \) is said to be starlike of order \( \beta \) \((0 \leq \beta < 1)\) in the unit disk \( U \) if and only if

\[
\Re\{zf'(z)/f(z)\} > \beta \quad (z \in U)
\]

for some \( \beta \) \((0 \leq \beta < 1)\). Further, a function \( f(z) \) belonging to the class \( A \) is said to be convex of order \( \beta \) \((0 \leq \beta < 1)\) in the unit disk \( U \) if and only if

\[
\Re\left\{1 + zf''(z)/f'(z)\right\} > \beta \quad (z \in U)
\]

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*\( \sum \) stands for \( \sum_{k=2}^{\infty} \) unless stated otherwise.
for some $\beta (0 \leq \beta < 1)$. We denote by $S^*(\beta)$ and $K(\beta)$ the subclasses of $A$ whose members satisfy (1.2) and (1.3), respectively. Then, it is well-known that $S^*(\beta) \subset S^*$, $K(\beta) \subset K$ for $0 < \beta < 1$, and that $S^*(0) \equiv S^*$, $K(0) \equiv K$ for $\beta = 0$.

Let $f \star g(z)$ denote the Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum b_kz^k,$$

then

$$f \star S(z) = z + \sum a_kb_kz^k.$$

By using the Hadamard product, Ruscheweyh [15] defined

$$D^\alpha f(z) = (z/(1 - z)^{\alpha + 1}) \ast f(z) \quad (\alpha \geq 1)$$

which implies that

$$D^n f(z) = \{z(z^{n-1}f(z))^{(n)/n!}$$

for $n \in N \cup \{0\}$, where $N = \{1, 2, 3, \ldots\}$. We note that $D^0 f(z) = f(z)$ and $D^k f(z) = zf'(z)$. The symbol $D^n f(z)$ was named by Al-Amiri [1] the $n$-th order Ruscheweyh derivative of $f(z)$. With the notation (1.4), Ruscheweyh [15] introduced the classes $K_n$ of functions $f(z)$ in $A$ satisfying the following condition

$$\Re\{D^{n+1} f(z)/D^n f(z)\} > 1/2 \quad (z \in U)$$

for $n \in N \cup \{0\}$, and he showed the basic property

$$K_{n+1} \subset K_n$$

for each $n \in N \cup \{0\}$, where $K_0 \equiv S^*(1/2)$ and $K_1 \equiv K$. Further, in the notation (1.5) a class $K_{-1}$ can be defined as the class of functions $f(z)$ in $A$ satisfying

$$\Re\{f(z)/z\} > 1/2 \quad (z \in U).$$

Since $K_0 \equiv S^*(1/2) \subset S^*$, Ruscheweyh's result (1.6) implies that $K_n \subset S^* \subset S$ for each $n \in N \cup \{0\}$.

Recently, by using the $n$-th order Ruscheweyh derivative of $f(z)$, Singh and Singh [18] introduced the subclass $R_n$ of $A$ whose members are characterized by the following condition

$$\Re\{D^{n+1} f(z)/D^n f(z)\} > n/(n + 1) \quad (z \in U)$$

for $n \in N \cup \{0\}$. We can immediately see that $R_0 \equiv S^*$ and $R_n \subset K_n$ for each $n \in N$. Hence $R_n$ is a subclass of $S^* \subset S$ for each $n \in N \cup \{0\}$. Further we can observe that $R_{n+1} \subset R_n$ for every $n \in N \cup \{0\}$.

In recent years, many classes defined by using the $n$-th order Ruscheweyh derivative of $f(z)$ were studied by Al-Amiri ([2], [3]), Bulboaca [4], Goel and Sohi ([6], [7]), and Owa ([10], [11]).
In this paper, we introduce the following classes $R(\alpha)$ by using the symbol $D^\alpha f(z)$.

**Definition 1.** We say that $f(z)$ is in the class $R(\alpha) (\alpha > 0)$, if $f(z)$ defined by

\[ f(z) = z - \sum akz^k \quad (a_k \geq 0) \]

satisfies the condition

\[ \Re \{ D^{\alpha+1} f(z) / D^\alpha f(z) \} > |x|/(\alpha + 1) \quad (z \in U) \]

for some $\alpha (\alpha \geq 0)$.

Since $D^1 f(z) = zf'(z)$ and $D^0 f(z) = f(z)$, we can see that $R(0) \equiv T^\alpha$ for $\alpha = 0$, which is the subclass of $S^\alpha$ consisting of functions $f(z)$ of the form (1.7) and was studied by Silverman [17]. $R(1)$ is the subclass of $K_1 \equiv K$ consisting of functions $f(z)$ of the form (1.7). Further, $R(n)$ is the subclass of $R_n$ consisting of functions $f(z)$ of the form (1.7).

2. Fractional calculus

Many essentially equivalent definitions of the fractional calculus that is, fractional derivatives and fractional integrals, have been given in the literature (cf. e.g., [5, Chapter 13], [8], [9], [14], [16], and [19, P. 28 et seq.]). We find it to be convenient to recall here the following definitions which were used recently by Owa [12].

**Definition 2.** The fractional integral of order $\alpha$ is defined by

\[ D_x^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(\zeta)d\zeta}{(z-\zeta)^{1+\alpha}}, \]

where $\alpha > 0$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

**Definition 3.** The fractional derivative of order $\alpha$ is defined by

\[ D_x^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{1+\alpha}}, \]

where $0 \leq \alpha < 1$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

**Definition 4.** Under the hypotheses of Definition 3, the fractional derivative of order $(n+\alpha)$ is defined by

\[ D_x^{n+\alpha} f(z) = D^n D_x^{\alpha} f(z)/dz^n, \]

where $0 \leq \alpha < 1$ and $n \in N \cup \{0\}$. 
With these definitions, recently, Owa [13] showed the following lemma.

**Lemma.** Let the function $f(z)$ be defined by (1.7). Then
\[
D^{n+\alpha} f(z) = z \left\{ D^{n+\alpha}_{z} \left( z^{n+\alpha-1} f(z) \right) \right\} / \Gamma(n + \alpha + 1)
\]
for $0 \leq \alpha < 1$, $n \in N \cup \{0\}$ and $z \in U$.

3. Distortion theorems

By using the lemma, we state and prove

**Theorem 1.** Let $n \in N \cup \{0\}$, $0 \leq \alpha < 1$ and the function $f(z)$ be defined by (1.7). Then $f(z)$ is in the class $R(n + \alpha)$ if and only if
\[
\sum \frac{(n + \alpha k + k)! \Gamma(n + \alpha + k)}{(k-1)!} a_k \leq \Gamma(n + \alpha + 2).
\]

**Equality holds for functions** \[ f(z) \text{ given by}
\]
\[
f(z) = z - \frac{\Gamma(n + \alpha + 2)(k-1)!}{(n + \alpha k + k)! \Gamma(n + \alpha + k)} z^k \quad (k \geq 2)
\]

**Proof.** Assume that the inequality (3.1) holds true and let $|z| = 1$. Then, by virtue of the lemma, we obtain that
\[
\left| \frac{D^{n+\alpha+1} f(z)}{D^{n+\alpha} f(z)} - 1 \right| = \left| \frac{\sum (1-k) \Gamma(n + \alpha + k)}{(n+\alpha+1)(k-1)!} a_k z^{k-1}}{\Gamma(n + \alpha + 1) - \sum \Gamma(n + \alpha + k)} - \frac{\Gamma(n + \alpha + k)}{(k-1)!} a_k z^k \right|
\]
\[
\leq \frac{\sum \Gamma(n + \alpha + k)}{(n+\alpha+1)(k-2)!} a_k \leq 1/(\alpha + 1).
\]

This shows that the values of $D^{n+\alpha+1} f(z)/D^{n+\alpha} f(z)$ lie in a circle centered at $w = 1$ whose radius is $1/(\alpha + 1)$. Consequently, we can see that the function $f(z)$ satisfies the condition (1.8), hence, $f(z) \in R(n + \alpha)$.

For the converse, assume that the function $f(z)$ belongs to the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then we get
\[
\Re \left\{ \frac{D^{n+\alpha+1} f(z)}{D^{n+\alpha} f(z)} \right\} = \Re \left\{ \frac{\Gamma(n + \alpha + 1) - \sum \frac{\Gamma(n + \alpha + k)}{(n+\alpha+1)(k-1)!} a_k z^{k-1}}{\Gamma(n + \alpha + 1) - \sum \frac{\Gamma(n + \alpha + k)}{(k-1)!} a_k z^k} \right\} > \alpha/(\alpha + 1)
\]
for $z \in U$. Choose values of $z$ on the real axis so that $D^{n+\alpha+1} f(z)/D^{n+\alpha} f(z)$ is real. Upon clearing the denominator in (3.3) and letting $z \to 1^-$ through real values, we can observe that
\[
\Gamma(n + \alpha + 1) - \sum \frac{\Gamma(n + \alpha + k + 1)}{(n + \alpha + 1)(k-1)!} a_k \geq \frac{\alpha}{\alpha + 1} \left\{ \Gamma(n + \alpha + 1) - \sum \frac{\Gamma(n + \alpha + k)}{(k-1)!} a_k \right\}
\]
which implies (3.1).

Finally we can show that the function \( f(z) \) given by (3.2) is an extremal function for the theorem. This completes the proof of Theorem 1.

**Corollary 1.** Let the function \( f(z) \) defined by (1.7) be in the class \( R(n+\alpha) \) for \( n \in \mathbb{N} \cup \{0\} \) and \( 0 \leq \alpha < 1 \). Then

\[
a_k \leq \frac{(k-1)!\Gamma(n+\alpha+2)}{(n+ak+k)\Gamma(n+\alpha+k)}
\]

for \( k \geq 2 \). The equality holds for the function \( f(z) \) of the form (3.2).

Applying Theorem 1, we prove

**Theorem 2.** Let the function \( f(z) \) defined by (1.7) be in the class \( R(n+\alpha) \) for \( n \in \mathbb{N} \cup \{0\} \) and \( 0 \leq \alpha < 1 \). Then

\[
|z| - (n + 2\alpha + 2)^{-1} |z|^2 \leq f(z) \leq |z| + (n + 2\alpha + 2)^{-1} |z|^2
\]

for \( z \in U \). The result is sharp.

**Proof.** Since \( f(z) \) belongs to the class \( R(n+\alpha) \), by using Theorem 1, we have

\[
(n + 2\alpha + 2)\Gamma(n + a + 2) \sum a_k \leq \sum \frac{(n + ak + k)\Gamma(n + \alpha + k)}{(k-1)!} a_k \leq \Gamma(n + \alpha + 2)
\]

which gives that \( \sum a_k \leq (n + 2\alpha + 2)^{-1} \). Hence we can see that

\[
|f(z)| \geq |z| - |z|^2 \sum a_k \geq |z| - (n + 2\alpha + 2)^{-1} |z|^2,
\]

\[
|f(z)| \leq |z| + |z|^2 \sum a_k \leq |z| + (n + 2\alpha + 2)^{-1} |z|^2
\]

for \( z \in U \).

Further, by taking the function

\[
f(z) = z - (n + 2\alpha + 2)^{-1} z^2
\]

we can prove that the result of the theorem is sharp.

**Corollary 2.** Let the function \( f(z) \) defined by (1.7) be in the class \( R(n+\alpha) \) for \( n \in \mathbb{N} \cup \{0\} \) and \( 0 \leq \alpha < 1 \). Then \( f(z) \) is included in a disk with its center at the origin and radius \( R \) given by \( R = (n + 2\alpha + 3)(n + 2\alpha + 2)^{-1} \).

**Theorem 3.** Let the function \( f(z) \) defined by (1.7) be in the class \( R(n+\alpha) \) for \( n \in \mathbb{N} \cup \{0\} \) and \( 0 \leq \alpha < 1 \). Then, for \( z \in U \),

\[
1 - (2 + \alpha)|\overline{z}|/(n + 2\alpha + 2) \leq |f'(z)| \leq 1 + (2 + \alpha)|\overline{z}|/(n + 2\alpha + 2).
\]

**Proof.** In view of Theorem 1, we have

\[
(n + 2\alpha + 2)\Gamma(n + \alpha + 2)(2 + \alpha)^{-1} \sum k a_k
\]

\[
\leq \sum \frac{(n + ak + k)\Gamma(n + \alpha + k)}{(k-1)!} a_k \leq \Gamma(n + \alpha + 2)
\]
which implies that
\begin{equation}
\sum k a_k \leq (2 + \alpha)(n + 2\alpha + 2)^{-1}.
\end{equation}
Consequently, by using (3.4), we have
\begin{align*}
|f'(z)| &\geq 1 - |z| \sum k a_k \geq 1 - (2 + \alpha)|z|/(n + 2\alpha + 2), \\
|f'(z)| &\leq 1 + |z| \sum k a_k \leq 1 + (2 + \alpha)|z|/(n + 2\alpha + 2)
\end{align*}
for \(z \in U\).

\textbf{Remark.} We have not been able to obtain a sharp estimate \(|f'(z)|\) in Theorem 3.

\textbf{Corollary 3.} Let the function \(f(z)\) defined by (1.7) be in the class \(R(n+\alpha)\) for \(n \in N \cup \{0\}\) and \(0 \leq \alpha < 1\). Then \(f'(z)\) is included in a disk with its center at the origin and radius \(R'\) given by \(R' = (n + 3\alpha + 4)(n + 2\alpha + 2)^{-1}\).

The following sharp estimation for \(|f'(z)|\) is due to Professor M. Obradovic.

\textbf{Theorem 3'.} Let the function \(f(z)\) defined by (1.7) be in the class \(R(n+\alpha)\) for \(n \in N \cup \{0\}\) and \(0 \leq \alpha < 1\). Then, for \(z \in U\),
\[1 - 2|z|/(n + 2\alpha + 2) \leq |f'(z)| \leq 1 + 2|z|/(n + 2\alpha + 2)\].

The result is sharp.

\textbf{Proof.} By using Theorem 1, we have
\[\Gamma(n + \alpha + 2) \leq \sum \frac{(n + ak + k)\Gamma(n + \alpha + k)}{(k - 1)!} a_k = \sum \frac{(n + ak + k)\Gamma(n + \alpha + k)}{k!} k a_k.\] (*)

Consider the expression
\[\frac{(n + ak + k)\Gamma(n + \alpha + k)}{k!} = \frac{(n + ak + k)(n + ak + k - 1) \ldots (n + \alpha + 2)\Gamma(n + \alpha + 2)}{k!}\]
for \(n = 0:\)
\[(ak + k)\Gamma(\alpha + k)/k! \geq (\alpha + 1)\Gamma(\alpha + 2)\]
for \(n \in N:\)
\[(n + ak + k)\Gamma(n + \alpha + k)/k! \geq (n + 2\alpha + 2)\Gamma(n + \alpha + 2)/2.\]

So, we can conclude that, for every \(n \in N \cup \{0\}\), we have
\[(n + ak + k)\Gamma(n + \alpha + k)/k! \geq (n + 2\alpha + 2)\Gamma(n + \alpha + 2)/2,\]
and then from (*) we obtain
\[ \sum k a_k \geq 2/(n + 2\alpha + 2). \]
The rest of the proof is as for Theorem 3.

Furthermore, that this estimation is the best possible is shown by the function
\[ f(z) = z - z^2/(n + 2\alpha + 2). \]

We derive some distortion inequalities for the fractional calculus of functions belonging to the class \( R(n + \alpha) \).

**Theorem 4.** Let the function \( f(z) \) defined by (1.7) be in the class \( R(n + \alpha) \) for \( n \in N \cup \{0\} \) and \( 0 \leq \alpha < 1 \). Then

\[
\begin{align*}
|D_z^{-\lambda} f(z)| &\geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{2|z|}{(2+\lambda)(n+2\alpha+2)} \right\} \\
\left|D_z^{-\lambda} f(z)\right| &\leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 + \frac{2|z|}{(2+\lambda)(n+2\alpha+2)} \right\}
\end{align*}
\]
for \( \lambda > 0 \) and \( z \in U \). The results in (3.5) and (3.6) are sharp.

**Proof.** Let
\[
F(z) = \Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda} f(z) = z - \sum \frac{k!\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} a_k z^k
\]
for \( \lambda > 0 \). Then we can observe that
\[
0 < k!\Gamma(2+\lambda)/\Gamma(k+1+\lambda) < 2/(2+\lambda)
\]
for \( \lambda > 0 \) and \( k \leq 2 \). Hence, with the help of (3.8) and Theorem 1, we have
\[
\begin{align*}
|F(z)| &\geq |z| - \frac{2}{2+\lambda} |z|^2 \sum a_k \geq |z| - \frac{2|z|^2}{(2+\lambda)(n+2\alpha+2)} \\
|F(z)| &\leq |z| + \frac{2}{2+\lambda} |z|^2 \sum a_k \geq |z| + \frac{2|z|^2}{(2+\lambda)(n+2\alpha+2)}
\end{align*}
\]
which give (3.5) and (3.6), respectively. Further, we can see that the results in (3.5) and (3.6) are sharp for the function \( f(z) \) defined by
\[ f(z) = z - \frac{z^2}{n + 2\alpha + 2}. \]

**Corollary 4.** Let the function \( f(z) \) defined by (1.7) be in the class \( R(n + \alpha) \) for \( n \in N \cup \{0\} \) and \( 0 \leq \alpha < 1 \). Then \( D_z^{-\lambda} f(z) \) is included in a disk with its center at the origin and radius \( R^{-\lambda} \) given by
\[ R^{-\lambda} = \frac{1}{\Gamma(2+\lambda)} \left\{ 1 + \frac{2}{(2+\lambda)(n+2\alpha+2)} \right\}. \]
where \( \lambda > 0 \).

Applying Theorem 3' to the function \( F(z) \), we have

**Theorem 5.** Let the function \( f(z) \) defined by (1.7) be in the class \( R(n+\alpha) \) for \( n \in \mathbb{N} \cup \{0\} \) and \( 0 \leq \alpha < 1 \). Then

\[
|D^\lambda_z f(z)| \geq \max \left\{ 0, \frac{|z|}{\Gamma(2 + \lambda)} \left( 1 - \lambda \frac{(2 + \lambda)z}{(n + 2\alpha + 2)} \right) \right\},
\]

\[
|D^\lambda_z f(z)| \leq \frac{|z|^\lambda}{\Gamma(2 + \lambda)} \left( 1 + \lambda + \frac{(2 + \lambda)z}{(n + 2\alpha + 2)} \right)
\]

for \( \lambda > 0 \) and \( z \in U \).

**Proof.** It is easy to see that the function \( F(z) \) defined by (3.7) is also the class \( R(n+\alpha) \). Hence, by means of Theorem 3' we get

\[
1 - 2|z|/(n + 2\alpha + 2) \leq F'(z) \leq 1 + 2|z|/(n + 2\alpha + 2)
\]

for \( z \in U \) which gives (3.9) and (3.10).

**Corollary 5.** Let the function \( f(z) \) defined by (1.7) be in the class \( R(n+\alpha) \) for \( n \in \mathbb{N} \cup \{0\} \) and \( 0 \leq \alpha < 1 \). Then \( D^\lambda_z f(z) \) is included in a disk with its center at the origin and radius \( R^\lambda \) given by

\[
R^\lambda = \frac{1}{\Gamma(2 + \lambda)} \left( 1 + \lambda + \frac{2 + \lambda}{n + 2\alpha + 2} \right), \quad \text{where } \lambda > 0.
\]

4. Closure theorems

**Theorem 6.** Let the functions

\[
f_i(z) = z \sum a_{k,i} z^k \quad (a_{k,i} \geq 0)
\]

be in the class \( R(n+\alpha) \) for \( n \in \mathbb{N} \cup \{0\} \), \( 0 \leq \alpha < 1 \) and every \( i = 1, 2, 3, \ldots, m \). Then the function \( h(z) \) defined by

\[h(z) = \sum_{i=1}^m c_i f_i(z) \quad (c_i \geq 0)\]

is also in the class \( R(n+\alpha) \), where \( \sum_{i=1}^m c_i = 1 \).

**Proof.** By means of the definition of \( h(z) \), we can write

\[
h(z) = z - \sum_{i=1}^m \left( \sum_{k=0}^\infty c_{k,i} z^k \right) z^k.
\]

Further, by virtue of Theorem 1, we have

\[
\sum_{k=0}^\infty \frac{(n + ak + k)\Gamma(n + \alpha + k)}{(k-1)!} a_{k,i} \leq \Gamma(n + \alpha + 2)
\]
for every \(i = 1, 2, 3, \ldots, m\). Hence we can see that
\[
\sum_{i=1}^{m} \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - 1)!} \left( \sum_{i=1}^{m} c_i a_k \right) \\
\sum_{i=1}^{m} c_i \left\{ \sum_{i=1}^{m} \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - 1)!} a_k \right\} \\
\leq \left( \sum_{i=1}^{m} c_i \right) \Gamma(n + \alpha + 2) = \Gamma(n + \alpha + 2)
\]
which implies that \(h(z)\) is in the class \(R(n + \alpha)\) with the aid of Theorem 1.

**Theorem 7.** Let \(f_1(z) = z\) and
\[
f_k(z) = z - \frac{(k - 1)!\Gamma(n + \alpha + k)}{(n + \alpha k + k)!\Gamma(n + \alpha + k)} z_k \quad (k \in N - \{1\})
\]
Then \(f(z)\) is in the class \(R(n + \alpha)\) for \(n \in N \cup \{0\}\) and \(0 \leq \alpha < 1\) if and only if it can be expressed in the form
\[
f_k(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z),
\]
where \(\lambda_k \geq 0\) for \(k \in N\) and \(\sum_{k=1}^{\infty} \lambda_k = 1\).

**Proof.** Suppose that
\[
f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = z - \sum_{k=1}^{\infty} \frac{(k - 1)!\Gamma(n + \alpha + 2)}{(n + \alpha k + k)!\Gamma(n + \alpha + k)} \lambda_k z^k.
\]
Then we obtain that
\[
\sum \left\{ \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - 1)!} \frac{(k - 1)!\Gamma(n + \alpha + 2)}{(n + \alpha k + k)!\Gamma(n + \alpha + k)} \lambda_k \right\} \leq \Gamma(n + \alpha + 2).
\]
This shows that \(f(z)\) belongs to the class \(R(n + \alpha)\) for \(n \in N \cup \{0\}\) and \(0 \leq \alpha < 1\). Again, by using Theorem 1.

For the converse, suppose that \(f(z)\) belongs to the class \(R(n + \alpha)\) for \(n \in N \cup \{0\}\) and \(0 \leq \alpha < 1\). Again, by using Theorem 1, we can observe that
\[
a_k \leq \frac{(k - 1)!\Gamma(n + \alpha + 2)}{(n + \alpha k + k)!\Gamma(n + \alpha + k)} \quad (k \in N - \{1\}).
\]

Now, setting
\[
\lambda_k = \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - 1)!\Gamma(n + \alpha + 2)} a_k \quad (k \in N - \{1\})
\]
and \(\lambda_1 = 1 - \sum \lambda_k\), we have the representation (4.3). This completes the proof of Theorem 7.
5. Modified Hadamard product

Let \( f(z) \) be defined by (1.7) and \( g(z) \) be defined by

\[
g(z) = z - \sum b_k z^k \quad (b_k \geq 0).
\]

Further, let \( f \ast g(z) \) denote the modified Hadamard product of \( f(z) \) and \( g(z) \), that is, \( f \ast g(z) = z - \sum a_k b_k z^k \).

**Theorem 8.** Let the functions \( f_i(z) \) defined by (4.1) be in the classes \( R(n_i + \alpha_i) \) for \( n_i \in N \cup \{0\} \), \( 0 \leq \alpha < 1 \) and each \( i = 1, 2, 3, \ldots, m \), respectively. Then the modified Hadamard product \( f_1 \ast f_2 \ast \cdots \ast f_m(z) \) belongs to the class \( R(n + \alpha) \), where \( n + \alpha = \underbrace{1}_{i=1} \leq m \to \text{Min}\{n_i + \alpha_i\} \).

**Proof.** Since \( f_i(z) \in R(n + \alpha_i) \) for each \( i = 1, 2, 3, \ldots, m \), in view of Theorem 1, we get

\[
a_{k,i} \leq 1/(n_i + 2\alpha + 2) \quad (i = 1, 2, 3, \ldots, m).
\]

Therefore we can show that

\[
\sum \left( n + \alpha \right) \Gamma(n + \alpha + k) \left( \prod_{i=1}^{m} a_{k,i} \right) \leq \frac{(n + 2\alpha + 2) \Gamma(n + \alpha + k)}{\prod_{i=1}^{m} (n_i + 2\alpha_i + 2)} \leq \Gamma(n + \alpha + 2)
\]

with the help of (5.1) and Theorem 1. Thus we have Theorem 8.

**Corollary 6.** Let the functions \( f_i(z) \) defined by (4.1) be in the same class \( R(n + \alpha) \) for \( n \in N \cup \{0\} \), \( 0 \leq \alpha < 1 \) and every \( i = 1, 2, 3, \ldots, m \). Then the modified Hadamard product \( f_1 \ast f_2 \ast \cdots \ast f_m(z) \) also belongs to the class \( R(n + \alpha) \).

6. Radii of starlikeness and convexity

We determine the radii of starlikeness and convexity of functions \( f(z) \) belonging to the class \( R(n + \alpha) \).

**Theorem 9.** Let the function \( f(z) \) defined by (1.7) be in the class \( R(n + \alpha) \) for \( n \in N \cup \{0\} \) and \( 0 \leq \alpha < 1 \). Then \( f(z) \) is starlike of order \( \beta \) \( (0 \leq \beta < 1) \) in the disk \( |z| < r_0 \), where

\[
r_0 = \inf_{k \in \mathbb{N} \setminus \{1\}} \left\{ \left( 1 - \beta \right) (n + \alpha + k) \Gamma(n + \alpha + k) \right\}^{1/(k-1)}
\]

with equality for the function \( f(z) \) given by (4.2).

**Proof.** It suffices to show that

\[
|z f'(z) / f(z) - 1| < 1 - \beta
\]
for \(|z| < r_0\). Now, we can observe that
\[
|zf'(z)/f(z) - 1| \leq \frac{\sum (k-1)a_k|z|^{k-1}}{1 - \sum a_k|z|^{k-1}} \leq 1 - \beta
\]
if and only if
\[
\sum ((k - \beta)/(1 - \beta))a_k|z|^{k-1} \leq 1.
\]
Hence, by virtue of Theorem 1, we need only find values of \(|z|\) for which
\[
\left(\frac{k - \beta}{1 - \beta}\right)|z|^{k-1} \leq \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - 1)!\Gamma(n + \alpha + 2)} \quad (k \geq 2),
\]
which will be true when \(|z| \leq r_0\). This completes the proof of the Theorem.

Finally, we have

**Theorem 10.** Let the function \(f(z)\) defined by (1.7) be in the class \(R(n+\alpha)\) for \(n \in \mathbb{N} \cup \{0\}\) and \(0 \leq \alpha < 1\). Then \(f(z)\) is convex of order \(\beta\) \((0 \leq \beta < 1)\) in the disk \(|z| < r_1\), where

\[
r_1 = \inf_{k \in \mathbb{N}-\{1\}} \left\{ \frac{(1 - \beta)(n + \alpha k + k)\Gamma(n + \alpha + k)}{k!\Gamma(n + \alpha + 2)} \right\}^{1/(k-1)}.
\]

**Proof.** Since \(f(z)\) defined by (1.7) is convex of order \(\beta\) if and only if \(zf'(z)\) is starlike of order \(\beta\), we can show that the proof of the Theorem follows the proof of Theorem 9, with \(a_k\) replaced by \(ka_k\).

**Corollary 7.** Let the function \(f(z)\) defined by (1.7) be in the class \(R(n+\alpha)\) for \(n \in \mathbb{N} \cup \{0\}\) and \(0 \leq \alpha < 1\). Then \(f(z)\) is univalent and starlike for \(|z| < r_2\), where

\[
r_2 = \inf_{k \in \mathbb{N}-\{1\}} \left\{ \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{k!\Gamma(n + \alpha + 2)} \right\}^{1/(k-1)}.
\]

**Proof.** By taking \(\beta = 0\) in Theorem 9, we have the Corollary.

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