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NECESSARY AND SUFFICIENT CONDITIONS FOR A SYSTEM OF EIGENFUNCTIONS AND ASSOCIATED FUNCTIONS OF A STURM-LIOUVILLE OPERATOR WITH DISCONTINUOUS COEFFICIENTS TU POSSESS THE BASIS PROPERTY*

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Abstract. In this paper we establish necessary and sufficient conditions which an arbitrary minimal and complete system of eigenfunctions and associated functions of a Sturm–Liouville operator with discontinuous coefficients must satisfy in order to have the basis property in the sense of V.A. Il'in [**1**].

Introduction

1. Consider the nonselfadjoint Sturm-Liouville operator

$$L(u) = -(p(x)u')' + q(x)u, \tag{1}$$

which is defined on a finite interval $G = (a, b)$ of the real axis. Let $x_0 \in (a, b)$ be a point of discontinuity of the coefficients of the operator (1). If we introduce the notation

$$p(x) = \begin{cases} p_1(x), & x \in (a, x_0), \\ p_2(x), & x \in (x_0, b), \end{cases}$$

then the following conditions are imposed on the coefficients:

$$\begin{aligned} &1) \quad p_1(x) = p_1 = \text{ const. } > 0, x \in (a, x_0]; p_2(x) = p_2 = \text{ const. } > 0, x \in [x_0, b). \\ &2) \quad q(x) \in L_p^{\text{loc}}(G), 1 < p < \infty; q(x) \text{ is a complex-valued function.} \end{aligned} \tag{2}$$

Definition 1. A complex-valued function $u_\lambda^0(x) \not\equiv 0$ is called an eigenfunction of the operator (1)-corresponding to the (complex) eigenvalue λ if it satisfies the following conditions:

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a) $u_\lambda^0(x)$ is absolutely continuous on any closed subinterval of G.

b) $u_\lambda^{0'}(x)$ is absolutely continuous on any closed subinterval of the half–open intervals (a, x_0) and $[x_0, b)$.

c) $u_\lambda^0(x)$ satisfies the equation

$$-p_1 u_\lambda^{0''}(x) + q(x) u_\lambda^0(x) = \lambda \cdot u_\lambda^0(x)$$

almost everywhere on (a, x_0), and the equation

$$-p_2 u_\lambda^{0''}(x) + q(x) u_\lambda^0(x)$$

almost everywhere on (x_0, b).

d) $u_\lambda^0(x)$ and $u_\lambda^{0'}$ satisfy the junction conditions

$$u_\lambda^0(x_0 - 0) = u_\lambda^0(x_0 + 0),\ p_1 u_\lambda^{0'}(x_0 - 0) = p_2 u_\lambda^{0'}(x_0 + 0)$$

at the point of discontinuity of the coefficients.

Definition 2. A complex-valued function $u_\lambda^i(x)$, $i \in N$, is called an i-th associated function of the operator (1) corresponding to the eigenfunction $u_\lambda^0(x)$ and the eigenvalue λ if it satisfies the following conditions:

a) Conditions a), b) and d) of Definition 1 hold for $u_\lambda^i(x)$.

b) $u_\lambda^i(x)$ satisfies the equation

$$-p_1 u_\lambda^{i''}(x) + q(x) u_\lambda^i(x) = \lambda \cdot u_\lambda^i(x) - u_\lambda^{i-1}(x)$$

almost everywhere on (a, x_0), and the equation

$$p_2 u_\lambda^{i''}(x) + q(x) = \lambda \cdot u_\lambda^i(x) - u_\lambda^{i-1}(x)$$

almost everywhere on (x_0, b).

We shall suppose that for every eigenvalue λ both the corresponding eigenfunctions $u_\lambda^0(x)$ and the first associated function $u_\lambda^1(x)$ exist (see p. 2, §4). Let $\{u_n^i(x) |\, n \in N,\ i = 0, 1\}$ be any minimal and complete system in ${}_2L_2(G)$ of eigenfunctions and associated functions of the operator (1), and let $\{\lambda_n |\, n \in N\}$ be the corresponding system of eigenvalues. Assuming the finite limit points of the set $\{\sqrt{\lambda_n} |\, n \in N\}$ do not exist, we can enumerate the numbers λ_n in order of nondecrease of $v_n = \left|\sqrt{\lambda_n}\right|$[1] Denote by $\{v_n^i(x) |\, n \in N,\ i = 0, 1\}$ the system of functions biorthogonally dual in $L_2(G)$ to $\{u_n^i(x) |\, n \in N,\ i = 0, 1\}$, i.e. such a system that $v_n^i(x) \in L_2(G)$ and

$$(u_n^i, v_m^i) \equiv \int_G u_n^i(x) \overline{v_m^j(x)} dx = \begin{cases} 1, & \text{if } n = m \text{ and } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

[1] If $\lambda_n = r_n \cdot e^{i\varphi n}$, then $\sqrt{\lambda_n} = \sqrt{r_n} e^{i\varphi n/2}$, where $-\pi/2 < \varphi_n \le 3\pi/2$.

Let $f(x)$ be an arbitrary function from the class $L_2(G)$, and let μ be a positive number. We can form the partial sum of expansion of $f(x)$ in biorthogonal series:

$$\sigma_\mu(x,f) = \sum_{\substack{1\le n\le\mu\\ i=0,1}} (f, v_n^i) u_n^i(x).$$

Following Il'in [**1**], we make the following definition.

Definition 3. The system of functions $\{u_n^i(x) |\, n \in N,\ i = 0,1\}$ has the basis property if for any function $f(x) \in L_2(G)$ and any compact subset K of G

$$\lim_{\mu\to+\infty} \|\sigma_\mu(x,f) - f(x\|)_{L_2(K)} = 0. \tag{B}$$

Our main result is the following.

THEOREM. *Let $\{u_n^i(x) |\, n \in N,\ i = 0,1\}$ be a minimal and complete system in $L_2(G)$ of eigenfunctions and associated functions of the operator* (1) *whose coefficients satisfy* (2), *and let $\{v_n^i(x) |\, n \in N,\ i = 0,1\}$ be the system of functions biorthogonally dual in $L_2(G)$ to $\{u_n^i(x)\}$. Let the eigenvalues λ_n satisfy the following condition: there exist constants A and B not depending of the numbers λ_n such that*

$$1) \qquad |\Im\sqrt{\lambda_n}| \le A, \qquad n \in N. \tag{3}$$

$$2) \qquad \sum_{\|\Re\sqrt{\lambda_n}|-\mu|\le 1} 1 \le B \ \text{ for any } \ \mu \ge 0, \tag{4}$$

where B does not depend on μ. Then the following statements are equivalent:

a) *The system of functions $\{u_n^i(x) |\, n \in N,\ i = 0,1\}$ has the basis property.*

b) *For any compact subset K of G there is a constant $C(K)$ not depending on $n \in N$ such that*

$$\|u_n^i\|_{L_2(K)} \cdot \|v_n^i\|_{L_2(G)} \le C(K),\ \ n \in N,\ \ i = 0,1. \tag{5}$$

2. This Theorem is an extension of the known result of II'in [**1**, Theorem 1] to the case of second order operator with discontinuous coefficients. In [**1**] Il'in found necessary and sufficient conditions which an arbitrary minimal and complete system of eigenfunctions and associated functions of a nonselfadjoint ordinary differential operator of any order with sufficiently (locally) smooth coefficients must satisfy, in order to have the basis property. In [**4**] we considered the case of the Schrödinger operator with complex-valued potential $g(x) \in L_p^{\mathrm{loc}}(G),\ 1 < p < +\infty$.

3. The Theorem is proved by a method based only on the mean-value formulas for the eigenfunctions and associated functions of the operator (1). This method was elaborated by V.A. Il'in in [**1**]-[**2**]. Applying it to our case, we had to solve two problems that turned out to be technical ones. First, it was impossible to use the notion of fundamental solution of the corresponding differential equations in obtaining the necessary estimates for eigenfitnctions and associated functions of the operator (1), as it was done in [**1**]–[**2**]. We solved this problem in [**5**]–[**6**]. Second, it turned out to be difficult to estimate the spectral function of the operator (1), even in the simplest case of the piecewise constant coefficient $p(x)$. This problem is considered in §2 of the present paper.

In §1 a list of the mean-value formulas and the estimates of eigenfunctions and associated functions of the operator (1) are given. In §3 the conditions (5) are proved to be sufficient, and in §4 they are proved to be necessary.

4. We thank Professor V.A. Il'in for stating the problem and for valuable discussions concerning numerous problems in the spectral theory of differential operators.

§1. Auxiliary results

1. The following estimates of eigenfunctions and associated functions of the operator (1), which are of independent interest, play an important role in the proof of the Theorem.

Let K be an arbitrary compact subset of G, and let R be a positive number less than the distance $\rho(K, \partial G)$ from K to the boundary ∂G of G. Let

$$K_R = \{x \in G |\, \rho(x, \overline{K}) \leq R\},$$

where $\overline{K}$ is the intersection of all closed subintervals of G containing the compact K.

If coefficients $p(x)$ and $q(x)$ satisfy the conditions (2) and eigenvalues λ_n[2] satisfy (3)-(4), then the following assertions hold.

LEMMA 1. *For any compact subset K of the interval G there is a constant $C(K, p, q)$ not depending on n and i such that*

$$\max_{x \in K} |u_n^i(x)| \leq C(K, p, q) \|u_n^i\|_{L_2(K_R)},\, n \in N,\ i = 0, 1, 2, \ldots \tag{6}$$

LEMMA 2. *For any compact subset K of the interval G there exists a constant $D(K, p, q)$ not depending on n and i such that*

$$\max_{x \in K} |u_n^{i-1}(x)| \leq D(K, p, q) |\sqrt{\lambda_n}| \cdot \|u_n^i\|_{L_{(K_R)}},\, n \in N,\, i \in N.$$

Lemmas 1 and 2 are proved in [**5**] (see Theorems 1, 2 and Remark 3) and [**6**] (see p. 1, §1 and p. 3, §3 in Chapter 5).

2. In order to give the mean-value formulas we introduce the functions $h = \rho_1(x, t)$ and $h = (x, t)$ defined by

$$\int\limits_{x-\rho_1(x,t)}^{x} \frac{d\tau}{\sqrt{p(\tau)}} = t, \qquad \int\limits_{x}^{x+\rho_2(x,t)} \frac{d\tau}{\sqrt{p(\tau)}} = t,$$

[2]Without loss of generality we shall assume that $\lambda_n \neq 0$, $n \in N$

where $x \in (a,b)$, $t \in [0, t_x]$, and t_x is a sufficiently small positive number. For a fixed $x \in (a,b)$ we have the inverse functions $t = \overline{\rho}_1(x,h)$ and $t = \overline{\rho}_2(x,h)$, which will be used in the following form:

$$\overline{\rho}_1(x, x-\xi) = \int_\xi^x \frac{d\tau}{\sqrt{p(\tau)}}, \ \ \overline{\rho}_2(x, \xi - x) = \int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}}.$$

3. Let $u_n^0(\xi)$ be the eigenfunction of the operator (1) corresponding to the eigenvalue λ_n. Let $x \in G$ and let t be a positive number such that the points $x - \rho_1(x,t)$ and $x + \rho_2(x,t)$ belong to G. The following mean-value formula for $u_n^0(\xi)$ holds[3]:

$$\begin{gathered}\sqrt{p(x-\rho_1(x,t))} \cdot u_n^0(x - \rho_1(x,t)) + \sqrt{p(x+\rho_2(x,t))} \cdot u_n^0(x + \rho_2(x,t)) = \\ = \left(\sqrt{p(x-0)} + \sqrt{p(x+0)}\right) \cdot u_n^0(x) \cdot \cos\sqrt{\lambda_n} \cdot t - \\ - \frac{1}{\sqrt{\lambda_n}} \cdot \int\limits_x^{x+\rho_2(x,t)} q(\xi) u_n^0(\xi) \sin\sqrt{\lambda_n}(\overline{\rho}_2(x, \xi - x) - t) d\xi - \\ - \frac{1}{\sqrt{\lambda_n}} \cdot \int\limits_{x-\rho_1(x,t)}^{x} q(\xi) u_n^0(\xi) \sin\sqrt{\lambda_n}(\overline{\rho}_1(x, x-\xi) - t) d\xi - \end{gathered} \tag{8}$$

lf $x - \rho_1(x,t) < x_0 \le x$, then the mean-value formula for $u_n^0(\xi)$ has the form

$$\begin{gathered}\sqrt{p_1} \cdot u_n^0(x - \rho_1(x,t) + \sqrt{p_2} \cdot u_n^0(x + \rho_2(x,t)) = \\ = 2\sqrt{p_2} \cdot u_n^0(x) \cos\sqrt{\lambda_n} t + (\sqrt{p_1} - \sqrt{p_2}) u_n^0(x_0) \cos\sqrt{\lambda_n}(\overline{\rho}_1(x, x - x_0) - t) - \\ - \frac{1}{\sqrt{\lambda_n}} \cdot \int\limits_x^{x+\rho_2(x,t)} q(\xi) u_n^0(\xi) \sin\sqrt{\lambda_n}(\overline{\rho}_2(x, \xi - x) - t) d\xi - \\ - \frac{1}{\sqrt{\lambda_n}} \cdot \int\limits_{x-\rho_1(x,t)}^{x} q(\xi) u_n^0(\xi) \sin\sqrt{\lambda_n}(\overline{\rho}_1(x, x-\xi) - t) d\xi - \end{gathered} \tag{9}$$

If $x < x_0 < x + \rho_2(x,t)$, then the following holds:

$$\begin{gathered}\sqrt{p_1} \cdot u_n^0(x - \rho_1(x,t) + \sqrt{p_2} \cdot u_n^0(x + \rho_2(x,t)) = \\ = 2\sqrt{p_1} \cdot u_n^0(x) \cos\sqrt{\lambda_n} t + (\sqrt{p_2} - \sqrt{p_1}) u_n^0(x_0) \cos\sqrt{\lambda_n}(\overline{\rho}_1(x, x_0 - x) - t) - \\ - \frac{1}{\sqrt{\lambda_n}} \cdot \int\limits_x^{x+\rho_2(x,t)} q(\xi) u_n^0(\xi) \sin\sqrt{\lambda_n}(\overline{\rho}_2(x, \xi - x) - t) d\xi - \\ - \frac{1}{\sqrt{\lambda_n}} \cdot \int\limits_{x-\rho_1(x,t)}^{x} q(\xi) u_n^0(\xi) \sin\sqrt{\lambda_n}(\overline{\rho}_1(x, x-\xi) - t) d\xi - \end{gathered} \tag{10}$$

[3]If $x \neq x_0$ then we assume that $x_0 \notin (x - \rho_1(x,t), x + \rho_2(x,t))$.

Equalities (8)-(10) are special case of the equalities (13)–(15) from [**5**].

4. Let $u_n^1(\xi)$ be the first associated function of the operator (1) corresponding to the eigenvalue λ_n and the eigenfunction $u_n^0(\xi)$. If $x \in (a,b)$ and t is a positive number such that the points $x-\rho_1(x,t)$ and $x+\rho_2(x,t)$ belong to G, then the following mean-value formula for $u_n^1(\xi)$ holds[4]:

$$\begin{aligned}
&\sqrt{p(x-\rho_1(x,t))}\cdot u_n^1(x-\rho_1(x,t))+\sqrt{p(x+\rho_2(x,t))}\cdot u_n^1(x+\rho_2(x,t))=\\
&=\left(\sqrt{p(x-0)}+\sqrt{p(x+0)}\right)\cdot u_n^1(x)\cdot\cos\left(\sqrt{\lambda_n}t\right)+1/2\cdot\left(\sqrt{p(x-0)}+\right.\\
&\left.+\sqrt{p(x+0)}\right)u_n^0(x)\cdot t\cdot\sin(\sqrt{\lambda_n}t)/\sqrt{\lambda_n}-\\
&\qquad-\frac{1}{\sqrt{\lambda_n}}\cdot\int\limits_x^{x+\rho_2(x,t)}q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_2(x,\xi-x)-t)d\xi-\\
&\qquad-\frac{1}{\sqrt{\lambda_n}}\cdot\int\limits_{x-\rho_1(x,t)}^{x}q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_1(x,x-\xi)-t)d\xi+\\
&+\frac{1}{2\lambda_n}\cdot\int\limits_x^{x+\rho_2(x,t)}q(\xi)u_n^0(\xi)(\overline{\rho}_2(x,\xi-x)-t)\cos\sqrt{\lambda_n}(\overline{\rho}_2(x,\xi-x)-t)d\xi+\\
&+\frac{1}{2\lambda_n}\cdot\int\limits_{x-\rho_1(x,t)}^{x}q(\xi)u_n^0(\xi)(\overline{\rho}_1(x,x-\xi)-t)\cos\sqrt{\lambda_n}(\overline{\rho}_1(x,x-\xi)-t)d\xi-\\
&\qquad-\frac{1}{2\lambda_n^{3/2}}\int\limits_x^{x+\rho_2(x,t)}q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_2(x,\xi-x)-t)d\xi-\\
&\qquad\frac{1}{\lambda_n^{3/2}}\int\limits_{x-\rho_1(x,t)}^{x}q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_1(x,x-\xi)-t)d\xi-
\end{aligned}\tag{11}$$

If $x-\rho_1(x,t)<x_0\le x$, then the mean-value formula for $u_n^1(\xi)$ has the form

$$\begin{aligned}
&\sqrt{p_1}\cdot u_n^1(x-\rho_1(x,t))+\sqrt{p_2}\cdot u_n^1(x+\rho_2(x,t))=\\
&=2\sqrt{p_2}\cdot u_n^1\cos\left(\sqrt{\lambda_n}t\right)+\sqrt{p_2}u_n^0\cdot t\sin\left(\sqrt{\lambda_n}t\right)/\sqrt{\lambda_n}+\\
&+(\sqrt{p_1}-\sqrt{p_2})\,u_n^1(x_0)\cos\sqrt{\lambda_n}(\overline{\rho}_1(x,x-x_0)-t)+\\
&+\left(1/2\sqrt{\lambda_n}\right)\cdot(\sqrt{p_1}-\sqrt{p_2})\,u_n^0(x_0)(\overline{\rho}_1(x,x-x_0)-t)\sin\sqrt{\lambda_n}(\overline{\rho}_1(x,x-x_0)-t)-\\
&-\frac{1}{\sqrt{\lambda_n}}\cdot\int\limits_x^{x+\rho_2(x,t)}q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_2(x,\xi-x)-t)d\xi-
\end{aligned}$$

[4] If $x\ne x_0$ then we assume that $x_0\notin(x-\rho_1(x,t),x+\rho_2(x,t))$.

$$
\begin{gathered}
-\frac{1}{\sqrt{\lambda_n}}\cdot\int\limits_{x-\rho_1(x,t)}^{x} q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_1(x,x-\xi)-t)d\xi+ \qquad (12)\\
+\frac{1}{2\lambda_n}\cdot\int\limits_{x}^{x+\rho_2(x,t)} q(\xi)u_n^0(\xi)(\overline{\rho}_2(x,\xi-x)-t)\cos\sqrt{\lambda_n}(\overline{\rho}_2(x,\xi-x)-t)d\xi+\\
+\frac{1}{2\lambda_n}\cdot\int\limits_{x-\rho_1(x,t)}^{x} q(\xi)u_n^0(\xi)(\overline{\rho}_1(x,x-\xi)-t)\cos\sqrt{\lambda_n}(\overline{\rho}_1(x,x-\xi)-t)d\xi-\\
-\frac{1}{2\lambda_n^{3/2}}\int\limits_{x}^{x+\rho_2(x,t)} q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_2(x,\xi-x)-t)d\xi-\\
\frac{1}{\lambda_n^{3/2}}\int\limits_{x-\rho_1(x,t)}^{x} q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_1(x,x-\xi)-t)d\xi.
\end{gathered}
$$

If $x<x_0<x+\rho_2(x,t)$ then the following holds:

$$
\begin{gathered}
\sqrt{p_1}\cdot u_n^1(x-\rho_1(x,t))+\sqrt{p_2}\cdot u_n^1(x+\rho_2(x,t))=\\
=2\sqrt{p_2}\cdot u_n^1\cos\left(\sqrt{\lambda_n}t\right)+\sqrt{p_2}u_n^0\cdot t\sin\left(\sqrt{\lambda_n}t\right)/\sqrt{\lambda_n}+\\
+\left(\sqrt{p_1}-\sqrt{p_2}\right)u_n^1(x_0)\cos\sqrt{\lambda_n}(\overline{\rho}_1(x,x-x_0)-t)+\\
+\left(1/2\sqrt{\lambda_n}\right)\cdot\left(\sqrt{p_1}-\sqrt{p_2}\right)u_n^0(x_0)(\overline{\rho}_1(x,x-x_0)-t)\sin\sqrt{\lambda_n}(\overline{\rho}_1(x,x-x_0)-t)-\\
-\frac{1}{\sqrt{\lambda_n}}\cdot\int\limits_{x}^{x+\rho_2(x,t)} q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_2(x,\xi-x)-t)d\xi-\\
-\frac{1}{\sqrt{\lambda_n}}\cdot\int\limits_{x-\rho_1(x,t)}^{x} q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_1(x,x-\xi)-t)d\xi+ \qquad (13)\\
+\frac{1}{2\lambda_n}\cdot\int\limits_{x}^{x+\rho_2(x,t)} q(\xi)u_n^0(\xi)(\overline{\rho}_2(x,\xi-x)-t)\cos\sqrt{\lambda_n}(\overline{\rho}_2(x,\xi-x)-t)d\xi+\\
+\frac{1}{2\lambda_n}\cdot\int\limits_{x-\rho_1(x,t)}^{x} q(\xi)u_n^0(\xi)(\overline{\rho}_1(x,x-\xi)-t)\cos\sqrt{\lambda_n}(\overline{\rho}_1(x,x-\xi)-t)d\xi-\\
-\frac{1}{2\lambda_n^{3/2}}\int\limits_{x}^{x+\rho_2(x,t)} q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_2(x,\xi-x)-t)d\xi-
\end{gathered}
$$

$$-\frac{1}{\lambda_n^{3/2}}\int\limits_{x-\rho_1(x,t)}^{x} q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}(\overline{\rho}_1(x,x-\xi)-t)d\xi.$$

Formula (11) can be derived from the formula (16) in [**5**] in the following way: First we introduce the substitution of variables $\overline{\rho}_2(x,\xi-x)=\tau$ and $\overline{\rho}_1(x,x-\xi)=\tau$ in the last two integrals on the right-hand side of (16), and then we use the formula (8) of the present paper. Computing all the interior integrals in the formulas (19)-(20) in [**5**], we obtain the formulas (12) and (13) respectively.

§2. Estimate of the spectral function of operator (1)

1. In this section we establish an estimate of the spectral function

$$\theta(x,y,\mu)=\sum_{\substack{1\le n\le u\\ i=0,1}} u_n^i(x)\cdot\overline{v_n^i(u)},\quad \mu>0, \tag{14}$$

of the operator (1), playing an extremely important role in proving that conditions (5) are sufficient.

Let K be any compact set of positive measure lying strictly within G, and let R_0 be a number such that $0<2R_0\max\{1,\sqrt{p_1},\sqrt{p_2}\}<\rho(K,\partial G)$. First suppose that $x_0\in K$, and define the following functions:

$$1)\quad t(x,y,\mu,R)=\begin{cases} a(x)\dfrac{\sin\mu\overline{\rho}_2(x,y-x)}{\overline{\rho}_2(x,y-x)}, & x\le y\le x+\rho_2(x,R),\\ a(x)\dfrac{\sin\mu\overline{\rho}_1(x,x-y)}{\overline{\rho}_1(x,x-y)}, & x-\rho_1(x,R)\le y\le x,\\ 0, a<y<x-\rho_1(x,R)\ \text{or}\ x+\rho_2(x,R)<y<b, \end{cases} \tag{15}$$

where $x\in K$, $R\in[R_0,2R_0]$, $a(x_0)=2/\pi\left(\sqrt{p_1}+\sqrt{p_2}\right)$, $a(x)=1/\pi\sqrt{p_1}$ if $x<x_0$, and $a(x)=1/\pi\sqrt{p_2}$ if $x_0<x$.

2) Let x belong to $K\cap(a,x_0]$. If there exists $R_1\in[R_0,2R_0)$ such that $x_0<x+p_2(x,R_1)$, then we set

$$t(x,y,\mu,R;x_0^-)=$$
$$\begin{cases} a^-(x)\cdot\dfrac{\sin\mu(\overline{\rho}_2(x_0,y-x_0)+\overline{\rho}_2(x,x_0-x))}{\overline{\rho}_2(x_0,y-x_0)+\overline{\rho}_2(x,x_0-x)},\\ \qquad x_0\le y\le x_0+\rho_2(x_0,R-\overline{\rho}_2(x_0,(x,x_0-x)),\\ a^-(x)\cdot\dfrac{\sin\mu(\overline{\rho}_1(x_0,x_0-y)+\overline{\rho}_2(x,x_0-x))}{\overline{\rho}_1(x_0,x_0-y)+\overline{\rho}_2(x,x_0-x)},\\ \qquad x_0+\rho_1(x_0,R-\overline{\rho}_2(x,x_0,(x,x_0-x))\le y,\le x_0\\ 0, a<y<x_0-\rho_1(x_0,R-\overline{\rho}_2(x,x_0-x))\ \text{or}\\ \qquad <x_0+\rho_2(x_0,R-\overline{\rho}_2(x,x_0-x)<y<b, \end{cases} \tag{16}$$

where $R \in [R_1, 2R_0]$, $a^-(x_0) = 0$, and

$$a^-(x) = \frac{\sqrt{p_2} - \sqrt{p_1}}{\pi\sqrt{p_1}\left(\sqrt{p_1} + \sqrt{p_2}\right)}, \quad \text{if } x < x_0.$$

Otherwise, we define $t(x, y, \mu, R; x_0^-) = 0$ for all $y \in G$, $\in (R_0, 2R_0]$.

3) Let x belong to $K \cap [x_0, b)$. If there is a number $R_2 \in [R_0, 2R_0)$ such that $x - \rho_1(x, R_2) < x_0$, then we set

$$t(x, y, \mu, R; x_0^+) = \begin{cases} a^+(x)\cdot\dfrac{\sin\mu(\overline{\rho}_1(x_0, x_0 - y) + \overline{\rho}_1(x, x - x_0))}{\overline{\rho}_1(x_0, x_0 - y) + \overline{\rho}_1(x, x - x_0)}, \\ \qquad\qquad x_0 - \rho_1(x_0, R - \overline{\rho}_1(x_0, (x, x - x_0)) \le y \le x_0, \\ a^+(x)\cdot\dfrac{\sin\mu(\overline{\rho}_2(x_0, y - x_0) + \overline{\rho}_1(x, x - x_0))}{\overline{\rho}_2(x_0, y - x_0) + \overline{\rho}_1(x, x - x_0)}, \\ \qquad\qquad x_0 \le y \le x_0 + \rho_2(x_0, R - \overline{\rho}_1(x, x - x_0)), \\ 0, a < y < x_0 - \rho_1(x_0, R - \overline{\rho}_1(x, x - x_0)) \text{ or} \\ \qquad\qquad < x_0 + \rho_2(x_0, R - \overline{\rho}_1(x, x - x_0) < y < b, \end{cases} \tag{17}$$

where $R \in [R_2, 2R_0]$, $a^+(x_0) = 0$, and

$$a^+(x) = \frac{\sqrt{p_1} - \sqrt{p_2}}{\pi\sqrt{p_2}\left(\sqrt{p_1} - \sqrt{p_2}\right)}, \quad \text{if } x_0 < x.$$

Otherwise, we define $t(x, y, \mu, R; x_0^+) = 0$ for $y \in G$, $R \in [R_0, 2R_0]$.

4) Finally, we set

$$\omega(x, y, \mu, R) = \begin{cases} t(x, y, \mu, R) - t(x, y, \mu, R; x_0^-), & \text{if } x \in K \cap (a, x_0], y \in G, \\ t(x, y, \mu, R) - t(x, y, \mu, R; x_0^+), & \text{if } x \in K \cap [x_0, b), y \in G \end{cases}$$

where $R \in [R_0, 2R_0]$.

If x_0 does not belong to K, then we define

$$\omega(x, y, \mu, R) = t(x, y, \mu, R),$$

where $x \in K$, $y \in G$, $R \in [R_0, 2R_0]$, and R_0 is such that $x_0 \notin K_{2R_0}$.

Now we state the mentioned estimate for the spectral function (14).

LEMMA 3. *If the coefficients of the operator* (1) *satisfy* (2)*, and its eigenvalues* λ_n *satisfy* (3)-(4)*, then the following estimate holds uniformly with respect to* $x \in K$ *and uniformly with respect to* $R \in [R_0, 2R_0]$[5]:

$$\int_G |\theta(x, y, \mu) - \omega(x, y, \nu_{[\mu]}, R)|^2 dy = O(1), \quad \mu \to +\infty. \tag{18}$$

[5]We denote by $[\mu]$ the integer part of μ

We prove this estimate in the remainder of the section.

2. It is sufficient to prove the estimate (18) for K being an arbitrary closed interval $[c,d] \subset G$. The most interesting case obtains for $x_0 \in K$.

First we shall suppose that x belongs to the compact $K^- = [c, x_0]$.

Compute the Fourier coefficients of the function (15) with respect to the system $\{u_n^0(y) | \, n \in N\}$:

$$t_n^0(x,\mu,R) \overset{\text{def}}{=\!=} \int\limits_G (x,y,\mu,R) u_n^0(y)dy =$$

$$a(x) \cdot \left(\int\limits_{x-\rho_1(x,t)}^{x} \frac{\sin \mu\overline{\rho}_1(x,x-y)}{\overline{\rho}_1(x,x-y)} u_n^0(y)dy + \int\limits_{x}^{x+\rho_2(x,t)} \frac{\sin \mu\overline{\rho}_2(x,y-x)}{\overline{\rho}_2(x,y-x)} \cdot u_n^0(y)dy \right).$$

If $x \in K^-$ is such that $x_0 < x + \rho_2(x,R)$, then by substitution $h = \overline{\rho}_1(x,x-y)$ $(h = \overline{\rho}_2(x,y-x))$ in the first (second) integral on the right-hand side we obtain

$$t_n^0(x,\mu,R) =$$
$$= a(x) \int\limits_0^R \frac{\sin \mu h}{h} \Big(\sqrt{p(x-\rho_1(x,h))} u_n^0(x-\rho_1(x,h)) + \sqrt{p(x+\rho_2(x,h))} \cdot$$
$$\cdot u_n^0(x+\rho_2(x,h)) \Big) dh = a(x) \cdot \int\limits_0^{\overline{\rho}_2(x,x_0-x)} \frac{\sin \mu h}{h} \Big(\sqrt{p_1} u_n^0(x-\rho_1(x,h)) +$$
$$+ \sqrt{p_1} \cdot u_n^0(x+\rho_2(x,h)) \Big) dh + a(x) \int\limits_{\overline{\rho}_2(x,x_0-x)}^{R} \frac{\sin \mu h}{h} \Big(\sqrt{p_1} \cdot u_n^0(x-\rho_1(x,h)) +$$
$$+ \sqrt{p_2} \cdot u_n^0)x+\rho_2(x,h)) \Big) dh.$$

If $x + \rho_2(x,R) \le x_0$, then $h \in [0,R]$ in the first integral on the right-hand side of the last equality, and the second one does not exists. In what follows we will be dealing with the points $x \in K^-$ of the first kind only, this case being more complex.

Now applying the formula (8) to the first integral, and the formula (10) to the second one, we obtain the equality

$$t_n^0(x,\mu,R) = u_n^0(x) \cdot \frac{2}{\pi} \cdot \int\limits_0^R \frac{\sin \mu h \cdot \cos \sqrt{\lambda_n} h}{h} dh +$$
$$+ \frac{\sqrt{p_2} - \sqrt{p_1}}{\pi \sqrt{p_1}} \cdot u_n^0(x_0) \cdot \int\limits_{(\overline{\rho}_2(x,x_0-x)}^{x} \frac{\sin \mu h}{h} \cos \sqrt{\lambda_n} (\overline{\rho}_2(x,x_0-x) - h) dh - \tag{19}$$

$$-\frac{a(x)}{\sqrt{\lambda_n}} \cdot \int_0^R \frac{\sin \mu h}{h} \left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi) u_n^0(\xi) \sin \sqrt{\lambda_n} \left(\left| \int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| - h \right) d\xi \right) dh.$$

Compute the Fourier coefficients of the function (15) with respect to the system $\{u_n^1(y) |\, n \in N\}$. Analogously to the previous case we have

$$t_n^1(x,\mu,R) \stackrel{\text{def}}{=\!=} \int_G (x,y,\mu,R) u_n^1(y) dy =$$

$$a(x) \cdot \int_0^{\overline{\rho}_2(x,x_0-x)} \frac{\sin \mu h}{h} \left(\sqrt{p_1} u_n^1(x - \rho_1(x,h)) \right) + \sqrt{p_1} \cdot u_n^1(x + \rho_2(x,h)) dh +$$

$$a(x) \cdot \int_{\overline{\rho}_2(x,x_0-x)}^R \frac{\sin \mu h}{h} \left(\sqrt{p_1} u_n^1(x - \rho_1(x,h)) \right) + \sqrt{p_2} u_n^1 \cdot (x + \rho_2(x,h)) dh.$$

Apply the formula (11) to the first integral, and the formula (13) to the second one. It follows that

$$\begin{aligned}
t_n^1(x,\mu,R) &= u_n^1(x) \cdot \frac{2}{\pi} \cdot \int_0^R \frac{\sin \mu h \cdot \cos \sqrt{\lambda_n} h}{h} dh + \\
&+ u_n^0(x) \frac{1}{\pi \sqrt{\lambda_n}} \cdot \int_0^R \sin \mu h \cdot \sin \sqrt{\lambda_n} h dh + \\
&+ \frac{\sqrt{p_2} - \sqrt{p_1}}{\pi \sqrt{p_1}} \cdot u_n^1(x_0) \cdot \int_{(\overline{\rho}_2(x,x_0-x)}^R \frac{\sin \mu h}{h} \cos \sqrt{\lambda_n} (\overline{\rho}_2(x,x_0-x) - h) dh + \\
&+ \frac{\sqrt{p_2} - \sqrt{p_1}}{\pi \sqrt{p_1}} \cdot u_n^1 \int_{\overline{\rho}_2(x,x_0-x)}^R \frac{\sin \mu h}{h} (\overline{\rho}_2(x,x_0-x) - h) \sin \sqrt{\lambda_n} \overline{\rho}_2(x,x_0-x) - h) dh - \\
&- \frac{a(x)}{\sqrt{\lambda_n}} \cdot \int_0^R \frac{\sin \mu h}{h} \left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi) u_n^1(\xi) \sin \sqrt{\lambda_n} \left(\left| \int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| - h \right) d\xi \right) dh + \qquad (20) \\
&+ \frac{a(x)}{2\lambda_n} \cdot \int_0^R \frac{\sin \mu h}{h} \left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi) u_n^0(\xi) \left(\left| \int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| - h \right) \cdot \right. \\
&\left. \cdot \cos \sqrt{\lambda_n} \left(\left| \int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| - h \right) d\xi \right) dh - \frac{a(x)}{2\lambda_n^{3/2}} \cdot \int_0^R \frac{\sin \mu h}{h} \cdot
\end{aligned}$$

$$\cdot\left(\int\limits_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}\left(\left|\int\limits_x^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|-h\right)d\xi\right)dh.$$

We remark that $t_n^0(x_0,\mu,R)$ and $t_n(x_0,\mu,R)$ are obtained using the formulas (8) and (11) only, and they have a somewhat different, simpler form than (19) and (20).

3. Compute the Fourier coefficients of the function (16) with respect to the system $\{u_n^0(y)|\, n\in N\}$:

$$t_n^0(x,\mu,R;x_0^-)\stackrel{\text{def}}{=\!=}\int\limits_G(x,y,\mu,R;x_0^-)u_n^0(y)dy=$$

$$=a^-(x)\cdot\int\limits_{x_0}^{x_0+\rho_2(x_0,R-\overline{\rho}_2(x,x_0-x)}\frac{\sin\mu(\overline{\rho}_2(x_0,y-x_0)+\overline{\rho}_2(x,x_0-x))}{\overline{\rho}_2(x_0,y-x_0)+\overline{\rho}_2(x,x_0-x)}dy+$$

$$+\,a^-(x)\cdot\int\limits_{(x_0,R-\overline{\rho}_2(x,x_0-x)}^{x_0}\frac{\sin\mu(\overline{\rho}_1(x_0,x_0-y)+\overline{\rho}_2(x,x_0-x))}{\overline{\rho}_1(x_0,x_0-y)+\overline{\rho}_2(x,x_0-x)}dy.$$

If we use the substitutions

$$h=\overline{\rho}_2(x_0,y-x_0)+\overline{\rho}_2(x,x_0-x),\;\; h=\overline{\rho}_1(x_0,x_0-y)+\overline{\rho}_2(x,x_0-x)$$

in the first and the second integral respectively, then we have

$$t_n^0(x,\mu,R;x_0^-)=$$

$$=a^-(x)\cdot\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\left(\sqrt{p_1}\cdot u_n^0\left(x_0-\rho_1\left(x_0,h-\int\limits_x^{x_0}\frac{d\tau}{\sqrt{p_1}}\right)\right)+\right.$$

$$\left.+\sqrt{p_2}\cdot u_n^0\left(x_0+\rho_2\left(x_0,h-\int\limits_x^{x_0}\frac{d\tau}{\sqrt{p_1}}\right)\right)\right).$$

Applying here the mean–value formula (8) with $x=x_0$ and $t=h-\overline{\rho}_2(x,x_0-x)$, we obtain the equality[6]

$$t_n^0(x,\mu,R;x_0^-)=\frac{\sqrt{p_2}-\sqrt{p_1}}{\pi\sqrt{p_1}}\cdot u_n^0(x_0)\cdot\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\cos\sqrt{\lambda_n}(h-\overline{\rho}_2(x,x_0-x))dh-$$

$$-\frac{a(x)}{\sqrt{\lambda_n}}\cdot\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\left(\int\limits_{x-\rho_1(x_0,h-\int)}^{x+\rho_2(x_0,h-\int)}q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}\left(\left|\int\limits_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\right.\right. \tag{21}$$

$$\left.\left.+\,\overline{\rho}_2(x,x_0-x)-h\right)d\xi\right)dh$$

[6]In what follows let $\int=\int\limits_x^{x_0}\frac{d\tau}{\sqrt{p_1}}$,

Finally, we compute the Fourier coefficients of the function (16) with respect to the system $\{u_n^1(y)|\, n \in N\}$. Proceeding as in the previous case, we obtain

$$t_n^1(x,\mu,R;x_0^-) \stackrel{\text{def}}{=\!=} \int\limits_G (x,y,\mu,R;x_0^-)u_n^1(y)dy =$$

$$= a^-(x)\cdot \int\limits_{\overline{\rho}_2(x,x_0-x)}^{R} \frac{\sin\mu h}{h}\left(\sqrt{p_1}\cdot u_n^1\left(x_0-\rho_1\left(x_0, h-\int\limits_x^{x_0}\frac{d\tau}{\sqrt{p_1}}\right)\right)+\right.$$

$$\left.+\sqrt{p_2}\cdot u_n^1\left(x_0+\rho_2\left(x_0,h-\int\limits_x^{x_0}\frac{d\tau}{\sqrt{p_1}}\right)\right)\right),$$

wherefrom, by formula (11) with $x=x_0$ and $t=h-\overline{\rho}_2(x,x_0-x)$, it follows that[6]

$$\begin{aligned}
t_n^1(x,\mu,R;x_0^-) &= \frac{\sqrt{p_2}-\sqrt{p_1}}{\pi\sqrt{p_1}}\cdot u_n^1(x_0)\cdot\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\cos\sqrt{\lambda_n}(h-\overline{\rho}_2(x,x_0-x))dh+\\
&+\frac{\sqrt{p_2}-\sqrt{p_1}}{2\pi\sqrt{p_1}\cdot\sqrt{\lambda_n}}u_n^0\cdot\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}(\overline{\rho}_2(x,x_0-x)-h)\sin\sqrt{\lambda_n}(\overline{\rho}_2(x,x_0-x)-h)dh-\\
&-\frac{a^-(x)}{\sqrt{\lambda_n}}\cdot\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\left(\int\limits_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}\left(\left|\int\limits_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\right.\right.\\
&\left.\left.+\overline{\rho}_2(x,x_0-x)-h\right)d\xi\right)dh+\frac{a^-(x)}{2\lambda_n}\cdot\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\cdot\left(\int\limits_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)u_n^0(\xi)\cdot\right.\\
&\cdot\left.\left(\left|\int\limits_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\int-h\right)\cos\sqrt{\lambda_n}\left(\left|\int\limits_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\int-h\right)d\xi\right)dh-\\
&-\frac{a^-(x)}{2\lambda_n^{3/2}}\cdot\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\left(\int\limits_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\right.\right.\\
&\left.\left.+\overline{\rho}_2(x,x_0-x)-h\right)d\xi\right)dh.
\end{aligned}\tag{22}$$

4. Denote

$$\omega(x,y,\mu,R;x_0^-)=t(x,y,\mu,R)-t(x,y,\mu,R;x_0),\tag{23}$$

where $x\in K^-$, $y\in G$ and $R\in[R_0,2R_0]$. By the equalities (19), (21), (20) and (22) the Fourier coefficients of this function with respect to the system $\{u_n^i(y)|\, n\in$

$N, i = 0, 1\}$ have the form

$$\omega_n^0(x,y,\mu,R;x_0^-) = t_n^0(x,y,\mu,R) - t_n^0(x,y,\mu,R;x_0) = u_n^0(x)\cdot\frac{2}{\pi}\cdot$$
$$\int\limits_0^R \frac{\sin\mu h\cos\sqrt{\lambda_n}h}{h}dh - \frac{a(x)}{\sqrt{\lambda_n}}\int\limits_0^R \frac{\sin\mu h}{h}\left(\int\limits_{x-\rho_1(x_0,h-\int)}^{x+\rho_2(x_0,h-\int)} q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}\,\cdot\right.$$
$$\left.\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| - h\right)d\xi\right)dh + \frac{a^-(x)}{\sqrt{\lambda_n}}\cdot\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\cdot \tag{24}$$
$$\cdot\left(\int\limits_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}\left(\left|\int\limits_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| + \overline{\rho}_2(x,x_0-x) - h\right)d\xi\right)dh,$$

$$\omega_n^1(x,y,\mu,R;x_0^-) = t_n^1(x,y,\mu,R) - t_n^1(x,y,\mu,R;x_0) =$$
$$= u_n^1(x)\cdot\frac{2}{\pi}\cdot\int\limits_0^R\frac{\sin\mu h\cos\sqrt{\lambda_n}h}{h}dh + u_n^0(x)\cdot\frac{1}{\pi\sqrt{\lambda_n}}\cdot\int\limits_0^R\sin\mu h\sin\sqrt{\lambda_n}hdh -$$
$$-\frac{a(x)}{\sqrt{\lambda_n}}\int\limits_0^R\frac{\sin\mu h}{h}\left(\int\limits_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| - h\right)d\xi\right)dh +$$
$$+\frac{a(x)}{2\lambda_n}\cdot\int\limits_0^R\frac{\sin\mu h}{h}\cdot\left(\int\limits_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi)u_n^0(\xi)\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| - h\right)\cos\sqrt{\lambda_n}\cdot\right.$$
$$\left.\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| - h\right)d\xi\right)dh - \frac{a(x)}{2\lambda_n^{3/2}}\cdot\int\limits_0^R\frac{\sin\mu h}{h}\cdot\left(\int\limits_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}\,\cdot\right. \tag{25}$$
$$\left.\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| - h\right)d\xi\right)dh + \frac{a^-(x)}{\sqrt{\lambda_n}}\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\left(\int\limits_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}\,\cdot\right.$$
$$\left.\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| + \overline{\rho}_2(x,x_0-x) - h\right)d\xi\right)dh - \frac{a^-(x)}{2\lambda_n}\cdot\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\cdot$$
$$\cdot\left(\int\limits_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| + \int -h\right)\cos\sqrt{\lambda_n}\cdot\right.$$

$$\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\int -h\right)d\xi\Bigg)dh+\frac{a^-(x)}{2\lambda_n^{3/2}}\cdot\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\cdot$$

$$\cdot\left(\int\limits_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)u_n^1(\xi)\sin\sqrt{\lambda_n}\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\overline{\rho}_2(x,x_0-x)-h\right)d\xi\right)dh.$$

5. Let $f(R)$ be an integrable function on $[R_0,2R_0]$, and let

$$S_{R_0}(f)=\frac{1}{R_0}\cdot\int_{R_0}^{2R_0}f(R)dR$$

be the S_{R_0}-transform of this function. If we denote

$$I^{\mu}_{\sqrt{\lambda_n}}(R)=\frac{2}{\pi}\cdot\int\limits_0^R\frac{\sin\mu h\cos\sqrt{\lambda_n}h}{h}dh,\quad \sqrt{\lambda_n}=\alpha_n+i\beta_n,$$

where $\alpha_n=\Re\sqrt{\lambda_n}$, $\beta_n=\Im\sqrt{\lambda_n}$, then the following holds:

$$S_{R_0}(I^{\mu}_{\sqrt{\lambda_n}}(R))=\delta^{\mu}_{\sqrt{\lambda_n}}+S_{R_0}(J^{\mu}_{\sqrt{\lambda_n}}(R)), \tag{26}$$

(see [**2**]). Here

$$\delta^{\mu}_{\sqrt{\lambda_n}}\overset{\text{def}}{=\!=}\frac{2}{\pi}\cdot\int\limits_0^R\frac{\sin\mu h\cos\sqrt{\lambda_n}h}{h}dh=\begin{cases}1, & \text{if } \alpha_n<\mu,\\ 1/2, & \text{if } \alpha_n=\mu,\\ 0, & \text{if } \alpha_n>\mu,\end{cases} \tag{27}$$

and $J^{\mu}_{\sqrt{\lambda_n}}(R)$ is a function such that

$$|S_{R_0}(J^{\mu}_{\sqrt{\lambda_n}}(R))|\le\begin{cases}C1(R_0,A), & \text{for every } \sqrt{\lambda_n} \text{ and } \mu,\\ \dfrac{C_2(R_0,A)}{|\mu-\alpha_n|^2}, & \text{if } |\mu-\alpha_n|>1,\end{cases} \tag{28}$$

the constants $C_1(R_0,A)$ and $C_2(R_0,A)$ not depending on $\sqrt{\lambda_n}$ and μ.

Denote

$$\omega_n^i(x,\mu;R_0;x_0^-)=\int\limits_G S_{R_0}(\omega(x,y,\mu,R;x_0))u_n^i(y)dy,\quad n\in N,\ \ i=0,1.$$

By Fubini's theorem we have

$$S_{R_0}(\omega_n^i(x,\mu;R_0;x_0^-))=\omega_n^i(x,\mu;R_0;x_0^-)\quad n\in N,\ \ i=0,1. \tag{29}$$

Multiply the equalities (24) and (25) by $\overline{v_n^0(y)}$ and $\overline{v_n^1(y)}$ respectively, denote by (24′) and (25′) the new equalities and apply the S_{R_0}-transform to (24′) and (25′). By (26) and (29) it follows that

$$
\begin{aligned}
&\sum_{i=0}^{1} \omega_n^i(x,\mu;R_0;x_0^-)\cdot\overline{v_n^i(y)} = \\
&= \sum_{i=0}^{1} u_n^i(x)\cdot\overline{v_n^i(y)}\delta^{\mu}_{\sqrt{\lambda_n}} + \sum_{i=0}^{1} u_n^i(x)\overline{v_n^i(y)}\cdot S_{R_0}(J^{\mu}_{\sqrt{\lambda_n}}(R)) + \\
&+ u_n^0(x)\cdot\overline{v_n^1(y)}\frac{1}{\pi\sqrt{\lambda_n}}\cdot S_{R_0}\left(\int_0^R \sin\mu h\sin\sqrt{\lambda_n}h\,dh\right) - \frac{a(x)}{\sqrt{\lambda_n}}\cdot\sum_{i=0}^{1}\overline{v_n^i(y)}S_{R_0}\cdot \\
&\cdot\left(\int_0^R \frac{\sin\mu h}{h}\left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi)u_n^i(\xi)\sin\sqrt{\lambda_n}\left(\left|\int_x^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| - h\right)d\xi\right)dh\right) + \\
&+ \frac{a^-(x)}{\sqrt{\lambda_n}}\cdot\sum_{i=0}^{1}\overline{v_n^i(y)}S_{R_0}\left(\int_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\left(\int_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)u_n^i(\xi)\sin\sqrt{\lambda_n}\cdot\right.\right. \\
&\left.\left.\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| + \int - h\right)d\xi\right)dh\right) + \frac{a(x)}{2\lambda_n}\cdot\overline{v_n^1(y)}S_{R_0}\left(\int_0^R\frac{\sin\mu h}{h}\cdot\right. \qquad (30)\\
&\left.\cdot\left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi)u_n^0(\xi)\left(\left|\int_x^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| - h\right)\cos\sqrt{\lambda_n}\left(\left|\int_x^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| - h\right)d\xi\right)dh\right) - \\
&- \frac{a^-(x)}{2\lambda_n}\cdot\overline{v_n^1(y)}S_{R_0}\left(\int_{\rho_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi)u_n^0(\xi)\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| + \int - h\right)\cdot\right.\right. \\
&\left.\left.\cdot\cos\sqrt{\lambda_n}\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| + \int - h\right)d\xi\right)dh\right) - \frac{a(x)}{2\lambda_n^{3/2}}\cdot\overline{v_n^1(y)}S_{R_0}\cdot \\
&\cdot\left(\int_0^R\frac{\sin\mu h}{h}\left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}\left(\left|\int_x^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| - h\right)d\xi\right)dh\right) + \\
&+ \frac{a^-(x)}{2\lambda_n^{3/2}}\cdot\overline{v_n^1(y)}S_{R_0}\left(\int_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\left(\int_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)u_n^0(\xi)\sin\sqrt{\lambda_n}\cdot\right.\right. \\
&\left.\left.\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right| + \int - h\right)d\xi\right)dh\right).
\end{aligned}
$$

According to (3)-(4), we have $\alpha_n = \mu_n$[7] if $\mu > A$. Without loss of generality we suppose that $\mu > A+1$. Summing from 1 to ∞, and' using (27), from (30) we obtain the following, for the time being formal, equality in $L_2(G)$ with respect to the variable y[8]:

$$\sum_{n,i} \omega_n^i(x,\mu;R_0;x_0^-)\cdot \overline{v_n^i(y)} =$$
$$= \sum_{\substack{\mu_n\le\mu\\ i=0,1}} u_n^i(x)\cdot\overline{v_n^i(y)} - \frac{1}{2}\sum_{\substack{\alpha_n=\mu\\ i=0,1}} u_n^i(x)\overline{v_n^i(y)} + \sum_{n,i} u_n^i(x)\cdot\overline{v_n^i(y)}\cdot S_{R_0}(J^{\mu}_{\sqrt{\lambda_n}}(R))+$$
$$+\frac{1}{\pi}\cdot\sum_{n=1}^{\infty} u_n^0(x)\cdot\overline{v_n^i(y)}\cdot S_{R_0}\left(\int_0^R \sin\mu h\frac{\sin\sqrt{\lambda_n}h}{\sqrt{\lambda_n}}dh\right) - a(x)\cdot\sum_{n,i}\overline{v_n^i(y)}\cdot S_{R_0}\cdot$$
$$\cdot\left(\int_0^R \frac{\sin\mu h}{h}\cdot\left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi)\frac{u_n^i(\xi)}{\sqrt{\lambda_n}}\cdot\sin\sqrt{\lambda_n}\cdot\left(\left|\int_x^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|-h\right)d\xi\right)dh\right)+$$
$$+a^-(x)\cdot\sum_{n,i}\overline{v_n^i(y)}\cdot S_{R_0}\cdot$$
$$\cdot\left(\int_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\left(\int_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)\frac{u_n^i(\xi)}{\sqrt{\lambda_n}}\cdot\sin\sqrt{\lambda_n}\cdot\left(\left|\int_x^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|-h\right)d\xi\right)dh\right)+$$
$$+\frac{a(x)}{2}\sum_{n=1}^{\infty}\overline{v_n^1(y)}S_{R_0}\cdot\left(\int_0^R\frac{\sin\mu h}{h}\cdot\right. \tag{31}$$
$$\left.\cdot\left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi)\frac{u_n^0(\xi)}{\lambda_n}\left(\left|\int_x^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|-h\right)\cos\sqrt{\lambda_n}\cdot\left(\left|\int_x^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|-h\right)d\xi\right)dh\right)-$$
$$-\frac{a^-(x)}{2}\sum_{n=1}^{\infty}\overline{v_n^1(y)}S_{R_0}\cdot\left(\int_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\cdot\left(\int_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)\frac{u_n^0(\xi)}{\lambda_n}\cdot\right.\right.$$
$$\left.\left.\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\int-h\right)\cos\sqrt{\lambda_n}\cdot\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\int-h\right)d\xi\right)dh\right)-$$
$$-\frac{a(x)}{2}\sum_{n=1}^{\infty}\overline{v_n^1(y)}S_{R_0}\cdot$$

---

[7] $\mu_n \stackrel{\text{def}}{=} |\Re\sqrt{\lambda_n}|$

[8] We denote $\sum\limits_{n=1}^{\infty}\sum\limits_{i=0}^{1}(\cdot)$ by $\sum\limits_{n,i}(\cdot)$

$$\cdot\left(\int\limits_0^R \frac{\sin\mu h}{h}\cdot\left(\int\limits_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi)\frac{u_n^0(\xi)}{\lambda_n^{3/2}}\cdot\sin\sqrt{\lambda_n}\cdot\left(\left|\int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}}\right|-h\right)d\xi\right)dh\right)+$$

$$+\frac{a^-(x)}{2}\sum_{n=1}^{\infty}\overline{v_n^1(y)}S_{R_0}\cdot$$

$$\cdot\left(\int\limits_0^R \frac{\sin\mu h}{h}\cdot\left(\int\limits_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi)\frac{u_n^0(\xi)}{\lambda_n^{3/2}}\cdot\sin\sqrt{\lambda_n}\cdot\left(\left|\int_{x_0}^\xi \frac{d\tau}{\sqrt{p(\tau)}}\right|-h\right)d\xi\right)dh\right),$$

where $x\in K^-$ is a point such that $R_0<\overline{\rho}(x,x_0-x)\le 2R_0$. We also suppose that the functions

$$f_i(R)=\int\limits_{\overline{\rho}(x,x_0-x)}^{R}(\cdot)$$

are equal to zero if $R\in[R_0,\overline{\rho}(x,x_0-x)]$. If point $x\in K^-$ is such that $\overline{\rho}(x,x_0-x)\le R_0$, then the series containing these functions do not appear on the right-hand side of (31).

6. Now we prove that all series on the right-hand side of (31) converge in $L_2(G)$, and L_2-norms of their sums, except for the first one, are bounded from above by constants not depending on $x\in[c,x_0]$ and $\mu\in(A+1,+\infty)$.

Fix the parameter μ. According to the condition (4), the first and the second series are finite, and their terms belong to $L_2(G)$. By (4)–(6) the following estimates for L_2-norm of the second sum are valid:

$$\left\|\sum_{\substack{\alpha_n=\mu\\ i=0,1}} u_n^i(x)\overline{v_n^i(y)}\right\|_{L_2(G)}\le\sum_{\substack{\alpha_n=\mu\\ i=0,1}}\max_{x\in K}|u_n^i(x)|\cdot\|\overline{v_n^i(y)}\|_{L_2(G)}\le$$

$$\le C(K,p,q)\cdot\sum_{\substack{\alpha_n=\mu\\ i=0,1}}\|u_n^i(x)\|_{L_2(K_{R_1})}\cdot\|\overline{v_n^i(y)}\|_{L_2(G)}\le 2C(K,p,q)\cdot C(K_{R_1})\cdot B,$$

where $0<R_1<\rho(K,\partial G)$.

For other series on the right-hand side of (31) we shall prove the stronger statement: Numerical series, consisting of the L_2-norms of terms of the initial series, are convergent for every $x\in K^-$, and their sums are bounded from above uniformly with respect to $x\in K^-$ and $\mu\in(A+1,+\infty)$.

For the series

$$\sum_{n,i}|u_n^i(x)|\cdot|S_{R_0}(J^{\mu}_{\sqrt{\lambda_n}}(R))|\cdot\|v_n^i\|_{L_2(G)}$$

and

$$\sum_{n=1}^{\infty} |u_n^0(x)| \cdot \left| S_{R_0} \left(\sin \mu h \frac{\sin \sqrt{\lambda_n} h}{\sqrt{\lambda_n}} dh \right) \right| \cdot \|v_n^1\|_{L_2(G)}$$

the mentioned statement was proved in [**4**, (28)–(30)].

In order to prove the convergence of series

$$\sum_{n,i} \left| S_{R_0} \left(\int_0^R \frac{\sin \mu h}{h} \cdot \left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi) \frac{u_n^i(\xi)}{\sqrt{\lambda_n}} \cdot \sin \sqrt{\lambda_n} \cdot \left(\left| \int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| - h \right) d\xi \right) dh \right) \right| \cdot \\ \cdot \|v_n^i\|_{L_2(G)} \tag{32}$$

we need some estimates for the integral

$$\begin{aligned} &\int_0^R \frac{\sin \mu h}{h} \cdot \left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi) u_n^i(\xi) \sin \sqrt{\lambda_n} \left(\left| \int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| - h \right) d\xi \right) dh = \\ &= \int_{x-\rho_1(x,h)}^{x} q(\xi) u_n^i(\xi) \left(\int_{\overline{\rho}_1(x,x-\xi)}^{R} \frac{\sin \mu h}{h} \cdot \sin \sqrt{\lambda_n} (\overline{\rho}_1(x,x-\xi) - h) dh \right) d\xi + \\ &+ \int_x^{x_0} q(\xi) u_n^i(\xi) \left(\int_{\overline{\rho}_2(x,\xi-x)}^{R} \frac{\sin \mu h}{h} \cdot \sin \sqrt{\lambda_n} (\overline{\rho}_2(x,\xi-x) - h) dh \right) d\xi + \\ &+ \int_{x_0}^{x+\rho_2(x,R)} q(\xi) u_n^i(\xi) \left(\int_{\overline{\rho}_2(x,\xi-x)}^{R} \frac{\sin \mu h}{h} \cdot \sin \sqrt{\lambda_n} (\overline{\rho}_1(x,\xi-x) - h) dh \right) d\xi \end{aligned} \tag{33}$$

and its S_{R_0}-transform. For all interior integrals on the right-hand side of (33) the estimates, analogous to the estimates (33)–(34) from [**4**], are valid. Proof of the estimates (33)–(34), [**4**], in the case of real numbers λ_n, may be found in [**3**, pp. 127–129]. That proof is also valid in our case of the complex λ_n, if we use condition (3) and the equality

$$\sin \sqrt{\lambda_n} t = \sin \alpha_n t \cdot \operatorname{ch} \beta_n t - i \cos \alpha_n t \cdot \operatorname{sh} \beta_n t.$$

It follows [**4**, (35)] that for integral (33) and its S_{R_0}-transform the estimates, analogous to the estimates (36)–(38) from [**4**], hold. Now the proof that series (32) converges and its sum is uniformly bounded from above proceeds exactly as in [**4**] for series (31) (see p.p. 102–103 therein).

Consider the series

$$\sum_{n=1}^{\infty} \left| S_{R_0} \left(\int_0^R \frac{\sin \mu h}{h} \cdot \left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi) \frac{u_n^0(\xi)}{\lambda_n} \left(\left| \int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| - h \right) \cos \sqrt{\lambda_n} d\xi \right) dh \right) \right| \cdot \\ \cdot \|v_n^1\|_{L_2(G)} \tag{34}$$

It is not difficult to see that for the S_{R_0}-transform of the integral

$$\int_0^R \frac{\sin \mu h}{h} \cdot \left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi) \frac{u_n^0(\xi)}{\sqrt{\lambda_n}} \left(\left| \int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| - h \right) \cos \sqrt{\lambda_n} \left(\left| \int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| - h \right) d\xi \right) dh$$

the estimates, analogous to the estimates (36)–(38) from [**4**], are valid. We estimate

$$\max_{x \in K_{2R_0}} |u_n^0(x)| \cdot \|v_n^1\|_{L_2(G)}$$

using (5) and (7):

$$\begin{aligned} \max_{x \in K_{2R_0}} |u_n^0(x)| \cdot \|v_n^1\|_{L_2(G)} &\le D(K_{2R_0}, p, q) |\sqrt{\lambda_n}| \|u_n^1\|_{L_2(K_{R_1})} \cdot \|v_n^1\|_{L_2(G)} \le \\ &\le |\sqrt{\lambda_n}| \cdot D(K_{2R_0}, p, q) \cdot C(K_{R_1}). \end{aligned} \tag{35}$$

By these facts one can prove the convergence of series (34) and uniform boundedness of its sum, using the same method as in the case of series (32).

The same is true for the series

$$\sum_{n=1}^\infty \left| S_{R_0} \left(\int_0^R \frac{\sin \mu h}{h} \left(\int_{x-\rho_1(x,h)}^{x+\rho_2(x,h)} q(\xi) \frac{u_n^0(\xi)}{\lambda_n^{3/2}} \sin \sqrt{\lambda_n} \left(\left| \int_x^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| - h \right) d\xi \right) dh \right) \right| \cdot$$
$$\cdot \|v_n^1\|_{L_2(G)}.$$

Now consider

$$\sum_{n,i} \left| S_{R_0} \left(\int_{\overline{\rho}_2(x,x_0-x)}^R \frac{\sin \mu h}{h} \left(\int_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi) \frac{u_n^i(\xi)}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n} \left(\left| \int_{x_0}^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| + \int - h \right) d\xi \right) dh \right) \right| \cdot$$
$$\cdot \|v_n^i\|_{L_2(G)}. \tag{36}$$

For the integral

$$\int_{\overline{\rho}_2(x,x_0-x)}^R \frac{\sin \mu h}{h} \left(\int_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)} q(\xi) \frac{u_n^i(\xi)}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n} \left(\left| \int_{x_0}^\xi \frac{d\tau}{\sqrt{p(\tau)}} \right| + \overline{\rho}_2(x, x_0 - x) - h \right) d\xi \right) dh =$$
$$= \int_{x_0}^{x_0+\rho_2(x_0,R-\int)} q(\xi) u_n^i(\xi) \left(\int_{\overline{\rho}_2(x,\xi-x)}^R \frac{\sin \mu h}{h} \cdot \sin \sqrt{\lambda_n} (\overline{\rho}_1(x, \xi - x) - h) dh \right) d\xi +$$
$$+ \int_{x_0-\rho_2(x_0,R-\int)}^{x_0} q(\xi) u_n^i(\xi) \left(\int_{\frac{x_0-(\xi+x)}{\sqrt{p_1}}}^R \frac{\sin \mu h}{h} \cdot \sin \sqrt{\lambda_n} \left(\frac{x_0 - (\xi + x)}{\sqrt{p_1}} - h \right) dh \right) d\xi$$

and its S_{R_0}-transform the estimates, analogous to the estimates (36)–(38) in [**4**] are valid. By these estimates one can prove the convergence of series (36) and uniform boundedness of its sum using the method mentioned above.

The corresponding assertions hold for the series

$$\sum_{n=1}^{\infty}\left|S_{R_0}\left(\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\left(\int\limits_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)}q(\xi)\frac{u_n^0(\xi)}{\lambda_n}\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\int-h\right)\cdot\right.\right.\right.$$

$$\left.\left.\left.\cdot\cos\sqrt{\lambda_n}\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\int-h\right)d\xi\right)dh\right)\right|\cdot\|v_n^i\|_{L_2(G)}$$

and

$$\sum_{n=1}^{\infty}\left|S_{R_0}\left(\int\limits_{\overline{\rho}_2(x,x_0-x)}^{R}\frac{\sin\mu h}{h}\cdot\right.\right.$$

$$\left.\left.\cdot\left(\int\limits_{x_0-\rho_1(x_0,h-\int)}^{x_0+\rho_2(x_0,h-\int)}q(\xi)\frac{u_n^0(\xi)}{\lambda_n^{3/2}}\sin\sqrt{\lambda_n}\left(\left|\int_{x_0}^{\xi}\frac{d\tau}{\sqrt{p(\tau)}}\right|+\overline{\rho}_2(x,x_0-x)-h\right)d\xi\right)dh\right)\right|\cdot$$

$$\cdot\|v_n^i\|_{L_2(G)}$$

(see also (35)).

Finally, we conclude that the equality (31) holds in the metric of $L_2(G)$, and L_2-norms of sums of all the series on the right-hand side of (31) are bounded from above uniformly with respect to $x\in K^-$ and uniformly with respect to $\mu\in(A+1,+\infty)$.

7. Notice that for every fixed $x\in K^-$ and μ the function $S_{R_0}(\omega(x,y,\mu,R;x_0^-))$ belongs to the class $L_2(G)$. This follows from the equality

$$S_{R_0}(\omega(x,y,\mu,R;x_0^-))=S_{R_0}(t(x,y,\mu,R))-S_{R_0}(t(x,y,\mu,R;x_0^-)),\tag{37}$$

and from the explicit form of the functions $S_{R_0}(t(x,y,\mu,R))$ and $S_{R_0}(t(x,y,\mu,R;x_0^-))$:

$$S_{R_0}(t(x,y,\mu,R;x_0^-))=$$

$$=\begin{cases}0,\quad a<y\le x-\rho_1(x,2R_0),\\ a(x)\cdot\dfrac{\sin\mu\overline{\rho}_1(x,x-y)}{\overline{\rho}_1(x,x-y)}\cdot\dfrac{2R_0-\overline{\rho}_1(x,x-y)}{R_0},\ \ x-\rho_1(x,2R_0)\le y\le\\ \qquad\qquad\qquad\qquad\qquad\le x-\rho_1(x,R_0),\\ a(x)\cdot\dfrac{\sin\mu\overline{\rho}_1(x,x-y)}{\overline{\rho}_2(x,x-y)},x-\rho_1(x,R_0)\le y\le x,\\ a(x)\cdot\dfrac{\sin\mu\overline{\rho}_2(x,y-x)}{\overline{\rho}_1(x,y-x)},x\le y\le x+\rho_2(x,R_0),\end{cases}\tag{38}$$

$$\begin{cases} a(x)\cdot\dfrac{\sin\mu\overline{\rho}_2(x,y-x)}{\overline{\rho}_2(x,y-x)}\cdot\dfrac{2R_0-\overline{\rho}_2(x,y-x)}{R_0}, & x+\rho_2(x,R_0)\le y\le \\ & \le x+\rho_2(x,2R_0), \\ 0, \quad x+\rho_2(x,2R_0)\le y<b. \end{cases}$$

$$S_{R_0}(t(x,y,\mu,R;x_0^-)) =$$

$$= \begin{cases} 0, \quad a<y\le x_0-\rho_1(x_0,2R_0-\int), \\ a^-(x)\dfrac{\sin\mu(\overline{\rho}_1(x_0,x_0-y)+\int)}{\overline{\rho}_1(x_0,x_0-y)+\int}\cdot\dfrac{2R_0-(\overline{\rho}_1(x_0,x_0-y+\int)}{R_0}, \\ \qquad x_0-\rho_1(x_0,2R_0-\int)\le y\le x_0-\rho_1(x_0,R_0-\int), \\ a^-(x)\dfrac{\sin\mu(\overline{\rho}_1(x_0,x_0-y)+\int)}{\overline{\rho}_2(x_0,x_0-y)+\int}, \quad x_0-\rho_1(x_0,R_0-\in)\le y\le x_0, \\ a^-(x)\cdot\dfrac{\sin\mu(\overline{\rho}_2(x_0,y-x_0)+\int)}{\overline{\rho}_1(x_0,y-x_0)+\int}, \quad x_0\le y\le x_0+\rho_2(x_0,R_0-\int), \\ a^-(x)\cdot\dfrac{\sin\mu(\overline{\rho}_2(x_0,y-x_0)+\int)}{\overline{\rho}_2(x_0,y-x_0)+\int}\cdot\dfrac{2R_0-\overline{\rho}_2(x,y-x_0)+\int}{R_0}, \\ \qquad x_0+\rho_2(x_0,R_0-\int)\le y\le x_0+\rho_2(x_0,2R_0-\int), \\ 0, \quad x_0+\rho_2(x_0,2R_0-\int)\le y<b. \end{cases} \tag{39}$$

By completeness of the system $\{u_n^i(\xi) \mid n\in N, i=0,1\}$ it follows that the following equality in the metric of $L_2(G)$ (with respect to y) holds:

$$S_{R_0}(\omega(x,y,\mu,R;x_0^-)) = \sum_{n,i}\omega_n^i(x,\mu;R_0;x_0^-)\cdot\overline{v_n^i(y)}. \tag{40}$$

Equalities (31) and (40) imply that for every fixed $x\in K^-$ and $\mu\in(A+1,+\infty)$ the following holds in $L_2(G)$:

$$S_{R_0}(\omega(x,y,\mu,R;x_0^-)) - \sum_{\substack{\mu_n\le\mu \\ i=0,1}} u_n^i(x)\cdot\overline{v_n^i(y)} = \mathcal{F}(x,y,\mu;R_0;x_0^-), \tag{41}$$

where $\mathcal{F}(x,y,\mu,R_0;x_o^-)$ is a function defined in an obvious way from (31). Also the estimate

$$\|\mathcal{F}(x,y,\mu;R_0;x_0^-)\|_{L_2(G)} = O(1), \quad \mu\to+\infty, \tag{42}$$

is valid uniformly with respect to $x\in K^-$.

8. From (15), (16), (23) and (37)–(39) we obtain the estimate

$$\int_G |S_{R_0}(\omega(x,y,\mu;R;x_0^-)) - \omega(x,y,\mu,R;x_0^-)|^2 dy = O(1), \quad \mu\to+\infty, \tag{43}$$

uniformly with respect to $R \in [R_0, 2R_0]$ and $x \in K^-$.

Finally, from (41)–(43), taking $\mu = \nu_{[\mu]}$[9], we obtain that

$$\left\| \sum_{\substack{\mu_n \le \nu_{[\mu]} \\ i=0,1}} u_n^i(x) \cdot \overline{v_n^i(y)} - \omega(x, y, \nu_{[\mu]}, R; x_0^-) \right\|_{L_2(G)} = O(1), \quad \mu \to +\infty \tag{44}$$

holds uniformly with respect to $x \in K^-$ and $R \in (R_0, 2R]$.

9. It remains to prove the estimate

$$\left\| \sum_{\substack{1 \le \| \le \| \mu \\ i=0,1}} u_n^i(x) \cdot \overline{v_n^i(y)} - \sum_{\substack{\mu_n \le \nu_{[\mu]} \\ i=0,1}} u_n^i(x) \cdot \overline{v_n^i(y)} \right\|_{L_2(G)} = O(1), \quad \mu \to +\infty \tag{45}$$

uniformly with respect to $x \in K^-$. For the proof on this estimate see [**2**, §3]. From (44)–(45) it follows that the estimate (18) is valid, uniformly with respect to $x \in K^-$ and $R \in [R_0, 2R_0]$.

10. In order to prove the estimate (18) on the compact $K^+ = [x_0, d]$ we define the following function:

$$\omega(x, y, \mu, R; x_0^+) = t(x, y, \mu, R) - t(x, y, \mu, R; x_0^+).$$

To compute the Fourier coefficients of this function it is necessary to use the mean-value formulas (8)–(9) and (11)–(12).

The other details in the proof of the estimate (18) on the compact K^+ are the same as in the case of the compact K^-.

This completes the proof of Lemma 3.

§3. Conditions (5) are sufficient

1. In this section we prove that conditions (5) are sufficient. Let K be any compact subset of the interval G containing the point x_0, and let $f(y)$ be an arbitrary function from $L_2(G)$. We introduce the function

$$S_\mu(x, f) = \frac{1}{\pi} \cdot \int_{x-r}^{x+r} \frac{sin\mu(x-y)}{x-y} f(y)dy,$$

where $x \in K$, and $r \in (0, \rho(K, \partial G)]$ is a fixed number not depending on $f(y)$. It is known that the function $S_\mu(x, f)$ differs on K from the partial sum of the order $[2\pi\mu/|G|]$ of the trigonometrical Fourier series of $f(x)$ by a function converging to 0 when $\mu \to +\infty$, uniformly with respect to $x \in K$.

[9] If $\mu \to +\infty$, then $\nu_{[\mu]} \to +\infty$

1. First we shall prove that equality (B) is valid on $K^- = [c, x_0]$. (It is sufficient to consider the case when K is a closed interval $[c,d]$.) By the Cauchy-Schwartz inequality from (18) we obtain the estimate

$$\int_G (\theta(x,y,\mu) - \omega(x,y,\nu_{[\mu]},R;x_0^-)) \cdot f(y)dy = O(1) \cdot \|f\|_{L_2(G)} \quad \mu \to +\infty, \qquad ((46))$$

uniformly with respect to $x \in K^-$, or

$$\sigma_\mu(x,f) - \int_G t(x,y,\nu_{[\mu]},R)f(y)dy + \int_G t(x,y,\nu_{[\mu]},R;x_0^-)f(y)dy =$$
$$= O(1) \cdot \|f\|_{L_2(G)}, \mu \to +\infty \qquad (47)$$

where R is a fixed number from the closed interval $[R_0, 2R_0]$.

Note that the second integral on the left-hand side of (47) does not appear if $x \in K^-$ is apoint such that $x + \rho_2(x,R) \le x_0$.

Let ε be any positive number. By completeness of the system $\{u_n^i \,|\, n \in N, i = 0,1\}$ there are numbers $n_0(\varepsilon) \in N$ and c_n^i, $i = 0,1$, $1 \le n \le n_0(\varepsilon)$, such that

$$\left\| f(y) - \sum_{oversetn=1 \rightarrow i=0,1}^{n_0(\varepsilon)} c_n^i \cdot u_n^i \right\|_{L_2(G)} \le \varepsilon \qquad (48)$$

Let

$$P_{n_0}(y) = \sum_{\substack{n=1\\ i=0,1}}^{n_0(\varepsilon)} c_n^i \cdot u_n^i.$$

Then $\sigma_\mu(y, P_{n_0}) = P_{n_0}(y)$, if $\mu \ge n_0(\varepsilon)$. Applying (47) to $f(y) - P_{n_0}(y)$, by (48) we obtain that[10]

$$\sigma_\mu(x,f) - P_{n_0}(x) - a(x) \cdot \int_{x-\rho_1(x,R)} \frac{\sin \nu_{[\mu]}\overline{\rho}_2(x,y-x)}{\overline{\rho}_2(x,y-x)}(f(y) - P_{n_0}(y))dy -$$
$$-a(x) \cdot \int_x^{x_0} \frac{\sin \nu_{[\mu]}\overline{\rho}_2(x,y-x)}{\overline{\rho}_2(x,y-x)}(f(y) - P_{n_0}(y))dy - a(x) \cdot$$
$$\cdot \int_{x_0}^{x+\rho_2(x,R)} \frac{\sin \nu_{[\mu]}\overline{\rho}_2(x,y-x)}{\overline{\rho}_2(x,y-x)}(f(y) - P_{n_0}(y))dy +$$
$$+ \int_G t(x,y,\nu_{[\mu]},R;x_0^-)(f(y) - P_{n_0}(y))dy = O(1) \cdot \varepsilon, \quad \mu \to +\infty,$$

[10] First we shall consider the case when $x \in K^-$ is a point such that $x_0 < x + \rho_2(x,R)$, i.e., $x \in K_1 \overset{\text{def}}{=\!=} (x_0 - \sqrt{p_1}R, x_0] \cap K^-$.

or

$$
\begin{gathered}
(\sigma_\mu(x,f)-f(x))+(1-\pi\sqrt{p_1}\cdot a(x))(f(y)-P_{n_0}(y)-\\
-\pi\sqrt{p_1}\cdot a(x)\left[\frac{1}{\pi}\cdot\int\limits_{x-\sqrt{p_1}R}^{x_0}\frac{\sin\nu_{[\mu]}\cdot p_1^{-1/2}\cdot|x-y|}{|x-y|}(f(y)-P_{n_0}(y))dy-\right.\\
\left.-(f(x)-P_{n_0}(x))\right]-a(x)\cdot\int\limits_{x_0}^{x+\sqrt{p_1}R}\frac{\sin\nu_{[\mu]}\overline{\rho}_2(x,y-x)}{\overline{\rho}_2(x,y-x)}(f(y)-P_{n_0}(y))dy=\\
=O(1)\cdot\varepsilon,\quad \mu\to+\infty,
\end{gathered}
\tag{49}
$$

holds uniformly with respect to $x\in K_1$.

3. Consider the function

$$
g_1(x,\mu)=\frac{1}{\pi}\cdot\int\limits_{x-\sqrt{p_1}R}^{x_0}\frac{\sin\nu_{[\mu]}\cdot p_1^{-1/2}\cdot|x-y|}{|x-y|}(f(y)-P_{n_0}(y))dy-(f(x)-P_{n_0}(x)),
$$

defined on K^-. If we define

$$
h_1(y)=\begin{cases} f(y)-P_{n_0}(y), & a<y\le x_0,\\ 0, & x_0<y<b,\end{cases}
$$

then we have the equality

$$
g_1(x,\mu)=S_{\nu_{[\mu]}/\sqrt{p_1}}(x,h_1)-h_1(x)
$$

on K^-. Note that $h_1(y)$ belongs to $L_2(HG)$, and the trigonometrical system is a basis in $L_2(G)$. Therefore for given ε there is a positive number $\mu_1(\varepsilon)$ such that

$$
\|g_1(x,\mu)\|_{L_2(K^-)}\le\varepsilon \tag{50}
$$

holds for every $\mu\ge\max\{A+1,n_0(\varepsilon),\mu_1(\varepsilon)\}$.

4. Introducing the function

$$
h_2(y)=\begin{cases} 0, & a<y\le x_0,\\ f(y)-P_{n_0}(y), & x_0<y<b,\end{cases}
$$

the integral

$$
g_2(x,\mu)\overset{\text{def}}{=\!=}\int\limits_{x_0}^{x+\rho_2(x,R)}\frac{\sin\nu_{[\mu]}\overline{\rho}_2(x,y-x)}{\overline{\rho}_2(x,y-x)}(f(y)-P_{n_0}(y))dy
$$

can be represented in the following form:

$$
\begin{gathered}
g_2(x,\mu)=\int\limits_{x_0}^{(x_0-\sqrt{p_2}\cdot\int)+\sqrt{p_2}\cdot R}\frac{\sin\nu_{[\mu]}p_2^{-1/2}(y-(x_0-\sqrt{p_2}\cdot\int))}{(y-(x_0-\sqrt{p_2}\cdot\int)}(f(y)-P_{n_0}(y))dy=\\
=\pi\sqrt{p_2}\cdot S_{\nu_{[\mu]}/\sqrt{p_1}}(x_0-\sqrt{p_2}\cdot\int,h_2),
\end{gathered}
\tag{51}
$$

where $x \in K_1$. Using the substitution $z = x_0 - (x_0 - x)\sqrt{p_2}/\sqrt{p_1}$, we obtain

$$\|S_{\nu_{[\mu]}/\sqrt{p_1}}(x_0 - \sqrt{p_2} \cdot \int, h_2)\|^2_{L_2(K_1)} = \frac{\sqrt{p_1}}{\sqrt{p_2}} \cdot \int\limits_{x_0 - \sqrt{p_2}R}^{x_0} |S_{\nu_{[\mu]}/\sqrt{p_1}}(z, h_2)|^2 dz \leq$$
$$\leq \frac{\sqrt{p_1}}{\sqrt{p_2}} \cdot \|S_{\nu_{[\mu]}/\sqrt{p_1}}(z, h_2)\|^2_{L_2(K_{\sqrt{p_2}R})}. \tag{52}$$

By the basis property of the trigonometrical system in $L_2(G)$, it follows that there exists a positive number $\mu_2(\varepsilon)$ such that

$$\|S_{\nu_{[\mu]}/\sqrt{p_1}}(z, h_2) - h_2(z)\|_{L_2(K_{\sqrt{p_2}R})} \leq \varepsilon$$

holds for every $\mu \geq \max\{A + 1, n_0(\varepsilon), \mu_2(\varepsilon)\}$. Therefrom, using the estimate (48), we obtain that

$$\|S_{\nu_{[\mu]}/\sqrt{p_1}}(z, h_2)\|_{L_2(K_{\sqrt{p_2}R})} \leq 2\varepsilon \tag{53}$$

Remark that $g_2(x, \mu) = 0$ for $x \in K \setminus K_1$. Thus by (51)–(53), the following estimate is valid:

$$\|g_2(x, \mu)\|_{L_2(K^-)} \leq 2\pi \cdot \sqrt[4]{p_1 p_2} \cdot \varepsilon \tag{54}$$

for every $\mu \geq \max\{A + 1, n_0(\varepsilon), \mu_2(\varepsilon)\}$.

5. Now we consider the integral

$$g_3(x, \mu) \stackrel{\text{def}}{=\!=} \int\limits_G t(x, y, \nu_{[\mu]}, R; x_0^-)(f(y) - P_{n_0}(y))dy =$$
$$= \sqrt{p_2} \cdot a^-(x) \cdot \int\limits_{x_0}^{x_0 + \sqrt{p_2}(R - \int)} \frac{\sin \nu_{[\mu]} \sqrt{p_2}^{-1/2}(y - x_0 + \sqrt{p_2} \cdot \int)}{(y - x_0 + \sqrt{p_2} \cdot \int)} (f(y) - P_{n_0}(y))dy + \tag{55}$$
$$+ \sqrt{p_2} \cdot a^-(x) \cdot \int\limits_{x_0 + \sqrt{p_1}(R - \int)}^{x_0} \frac{\sin \nu_{[\mu]} \sqrt{p_1}^{-1/2}(x_0 - y + \sqrt{p_1} \cdot \int)}{(x_0 - y + \sqrt{p_1} \cdot \int)} (f(y) - P_{n_0}(y))dy,$$

defined on K_1 (see (16)). Using the functions $h_1(y)$ and $h_2(y)$, we can rewrite (55) in the form

$$g_3(x, \mu) = \sqrt{p_2} \cdot a^-(x) \cdot \int\limits_{(x_0 - \sqrt{p_2} \cdot \int) - \sqrt{p_2}R}^{(x_0 - \sqrt{p_2} \cdot \int) + \sqrt{p_2}R} \frac{\sin \nu_{[\mu]} \sqrt{p_2}^{-1/2} |y - (x_0 - \sqrt{p_2} \cdot \int)|}{|y - (x_0 - \sqrt{p_2} \cdot \int)|} h_2(y)dy +$$
$$+ \sqrt{p_1} \cdot a^-(x) \cdot \int\limits_{(x_0 + \sqrt{p_1} \cdot \int) - \sqrt{p_1}R}^{(x_0 + \sqrt{p_1} \cdot \int) + \sqrt{p_1}R} \frac{\sin \nu_{[\mu]} \sqrt{p_1}^{-1/2} |y - (x_0 + \sqrt{p_1} \cdot \int)|}{|y - (x_0 + \sqrt{p_1} \cdot \int)|} h_1(y)dy =$$
$$= \pi \sqrt{p_2} a^-(x) \cdot S_{\nu_{[\mu]}/\sqrt{p_2}}(x_0 - \sqrt{p_2} \cdot \int, h_2) -$$
$$- \pi \sqrt{p_1} a^-(x) \cdot S_{\nu_{[\mu]}/\sqrt{p_1}}(x_0 + \sqrt{p_1} \cdot \int, h_1).$$

Proceeding as in the case of the integral $g_2(x,\mu)$, it can be shown that there exists $\mu_3(\varepsilon)>0$ such that the estimate

$$\|g_3(x,\mu)\|_{L_2(K^-)} \leq 2\frac{(\sqrt[4]{p_1p_2}+\sqrt{p_1})|\sqrt{p_1}-\sqrt{p_2}|}{\sqrt{p_1}(\sqrt{p_1}+\sqrt{p_2})}\cdot\varepsilon \tag{56}$$

holds for every $\mu \geq \max\{A+1, n_0(\varepsilon), \mu_3(\varepsilon)\}$.

6. If $x \in K \setminus K_1$, then (49) takes the simpler form

$$\begin{gathered}(\sigma_n(x,f)-f(x))+(1-\pi\sqrt{p_1}a(x))(f(x)-P_{n_0}(x))- \\ -\pi\sqrt{p_1}a(x)\cdot(S_{\nu_{[\mu]}/\sqrt{p_1}}(x,f-P_{n_0})-(f(x)-P_{n_0}(x)))=O(1)\cdot\varepsilon,\ \ \mu\to+\infty.\end{gathered} \tag{57}$$

According to (48), (50), (54) and (56), from (49) and (57) we finally obtain that

$$\lim_{\mu\to+\infty}\|\sigma_\mu(x,f)-f(x)\|_{L_2(K^-)}=0.$$

The proof of the equality (B) on $K^+=[x_0,d]$ is analogous. Thus conditions (5) are sufficient.

§4. Conditions (5) are necessary

1. In §1, [**1**], the following statement was proved: Let $\{u_n(x)|\, n\in N\}$ be any minimal and complete system in $L_2(G)$ and let $\{v_n(x)|\, n\in N\}$ be the corresponding system biorthogonally dual in $L_2(G)$ to $\{u_n(x)|\, n\in N\}$. If $\{u_n(x)|\, n\in N\}$ has the basis property, then the conditions (5) are satisfied. Note that $\{u_n(x)|\, n\in N\}$ is not necessarily connected with any operator.

From this result it follows that the conditions (5) are necessary in our special case.

This terminates the proof of the Theorem.

2. Our theorem has been proved assuming that for every eigenvalue of the operator (1) there exist an eigenfunction and associated function of the first order. But this theorem is also valid if a finite sequence of associated functions corresponds to every eigenvalue. One can prove this generalization, using the mean-value formulas obtained in [**5**], and the technique worked out in [**1**]–[**2**].

REFERENCES

[1] В.А. Ильин, *Необходимые и достаточные усцловия базисности равношодимости с тригонометрическим рядом спектральных разложений I*, Диференц. уравнения **16**(1980), 771–794.

[2] В.А. Ильин, *Необходимые и достаточные усцловия базисности равношодимости с тригонометрическим рядом спектральных разложений II*, Диференц. уравнения **16**(1980), 980–1009.

[3] I. Joó, N. Lažetić, *Estimate for the difference of the second derivative of partial sums arising by non-negative selfadjoint extensions of Sturm-Liouville operators*, Acta Math. Acad. Sci. Hungar. **38**(1981), 117–138.

[4] Н. Лажетич, *О шодимости разложений по собственным и присоединенным функциям несамосопряженного оператора типа Штурма-Лиувлля с разрывными коэффициентами*, Publ. Inst. Math. (Beograd) (N.S.) **30**(**44**)(1981), 89–109.

[5] Н. Лажетич, *Оценки собственных и присоединенных функций оператора Штурма-Лиувлля с разрывными коэффициентами*, *I* Bull. Acad. Serbe Sci. LXXXIV, Sci. Math. **13**(1984), 75–101.

[6] Н. Лажетич, *Оценки собственных и присоединенных функций оператора Штурма-Лиувлля с разрывными коэффициентами*, *II* Bull. Acad. Serbe Sci. LXXXVIII, Sci. Math. **14**(1985), 21–47.

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