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JUSTIFCATION OF THE AVERAGING METHOD FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH MAXIMUMS

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Abstract. The averaging method is justified for a class of functional-differential equations with maximums.

In recent years the theory of functional-differential equations with maximums have been developed [**1**], [**2**], in connection with their applications to various automatic control problems.

The development of effective approximation methods for such equations is of great interest since their solution in closed form is impossible even in the linear case.

The present paper is devoted to the justification of the averaging method for an initial value problem associated to a vector functional-differential equation of neutral type with maximums. In applications the maximum arises when the control law corresponds to the maximal deviation of the regulated quantity. If the control law takes into account also the maximal velocity of deviation of this quantity then the process is governed by a neutral type equation with maximums.

Consider the system of functional-differential equations

$$\begin{aligned} \dot{x}(t) &= \varepsilon X(t, x(t), \max\{x(s) : s \in [t-h, t]\}, \\ &\quad \max\{\dot{x}(s) : s \in [t-h, t]\}, \qquad t > 0, \\ x(t) &= \varphi(t), \dot{x}(t) = \dot{\varphi}(t), \qquad -h \le t \le 0, \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$, h is a positive constant

$$\begin{aligned} &\max\{x(s) : s \in [t-h, t]\} = \\ &= (\max\{x_1(s) : s \in [t-h, t]\}, \dots, \max\{x_n(s) : s \in [t-h, t]\}), \\ &\max\{\dot{x}(s) : s \in [t-h, t]\} = \\ &= (\max\{\dot{x}_1(s) : s \in [t-h, t]\}, \dots, \max\{\dot{x}_n(s) : s \in [t-h, t]\}), \end{aligned}$$

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$\varphi(t)$ is the initial function and $\varepsilon > 0$ is a small parameter.

Suppose that the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t,x,x,0)dt = \overline{X}(x) \tag{2}$$

exists. Then the averaged first approximation system is

$$\dot{\xi}(t) = \varepsilon \overline{X}(\xi(t)), \xi(0) = x(0). \tag{3}$$

Note that if $x = (x_1, \dots, x_n)$ then $\|x\| = \left[\sum_{i=1}^n x_i^2\right]^{1/2}$ by definition.

The following theorem gives conditions for proximity between the solutions $x(t)$ and $\varphi(t)$ of the initial value problems (1) and (3).

Theorem. *Let the following conditions be fulfilled:*

1. *The function $X(t,x,y,z)$ is continuous in the domain $\Omega(t,x,y,z) = \Omega(t) \times \Omega(x) \times \Omega(y) \times \Omega(z)$, where $\Omega(t) = [0,\infty)$, $\Omega(x) \equiv \Omega(y)$ and $\Omega(z)$ are domains in $\mathbf{R}^n$. The function $\varphi(t)$ is continuous and takes values in $\Omega(z)$ for $t \in [-h, 0]$, $\varphi(t) \in \Omega(x)$.*

2. *The function $X(t,x,y,z)$ satisfies the inequalities*

$\|X(t,x,y,z)\| \le M$, $\|X(t,x,y,z) - X(t',x',y',z')\| \le \lambda(\|x-x'\| + \|y-y'\| + \|z-z'\|)$ in the domain $\Omega(t,x,y,z)$, where M and λ are positive constants.

3. *The limit (2) exists for each $x \in \Omega(x)$. The functon $\overline{X}(x)$ is continuous in $\Omega(x)$.*

4. *For each $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon =$const > 0 the initial value problem (1) has a unique continuous solution $x(t)$ on the interval $0 \le t \le L\varepsilon^{-1}$.*

5. *For each $\varepsilon \in (0, \varepsilon_0]$ the intial value problem (3) has a unique continuous solution $\xi(t)$, such that $\xi(t)$ belongs to the domain $\Omega(x)$ together with its ρ-neighborhood for $0 \le t \le L\varepsilon^{-1}$ ($\rho =$const > 0).*

Then for each $\eta > 0$ and $L > 0$ there exists $\varepsilon_0 = \varepsilon_0(\eta, L) > 0$ such that $\|x(t) - \xi(t)\| < \eta$ for $0 < \varepsilon \le \varepsilon_0$ and $O \le t \le L\varepsilon^{-1}$.

Proof. The solutions of (1) and (3) can be represented in the form

$$\begin{aligned} &x(t) = x(0) + \\ &+ \varepsilon \int_0^t X(\theta, x(\theta), \max\{x(s) : s \in [\theta - h, \theta]\}, \max\{\dot{x}(s) : s \in [\theta - h, \theta]\})d\theta, \\ &t > 0, \ \dot{x}(t) = \varphi(t), \ \dot{x}(t) = \dot{\varphi}(t), \qquad -h \le t \le 0, \quad \text{and} \end{aligned} \tag{4}$$

$$\xi(t) = \xi(0) + \varepsilon \int_0^t \overline{X}(\xi(theta))d\theta. \tag{5}$$

Subtracting (5) from (4) one obtains

$$\|x(t)-\xi(t)\| \le \varepsilon \Bigg\| \int\limits_0^t [X(\theta), x(\theta), \max\{x(s) : s\in[\theta-h,\theta]\},$$

$$\max\{\dot{x}(s) : s\in[\theta-h,\theta]\}) - \overline{X}(\xi(\theta))]d\theta \Bigg\| \le$$

$$\le \varepsilon \int\limits_o^t \|X(\theta, \max\{x(s) : s\in[\theta-h,\theta]\}+ \tag{6}$$

$$+\,\varepsilon \Bigg\| \int\limits_0^t [X(\theta), \xi(\theta), \xi(\theta), 0) - \overline{X}(\xi(\theta))]d\theta \Bigg\| \equiv I_1 + I_2.$$

The following estimations for $0 \le t \le L\varepsilon^{-1}$ hold true in view of the conditions of the theorem:

$$I_1 = \Bigg\| \int\limits_0^t [X(\theta, x(\theta), \max\{x(s) : s\in[\theta-h,\theta]\},$$

$$\max\{\dot{x}(s) : s\in[\theta-h,\theta]\}) - X(\theta,\xi(\theta),\xi(\theta),0)]d\theta \Bigg\| \le$$

$$\le \varepsilon\lambda \int\limits_0^t \big[\|x(\theta)-\xi(\theta)\| + \|\max\{x(s) : s\in[\theta-h,\theta]\} - \xi(\theta)\| +$$

$$+\|\max\{\dot{x}(s) : s\in[\theta-h,\theta]\}\|\big]d\theta < \varepsilon\lambda \int\limits_0^t \|x(\theta)-\xi(\theta)\|d\theta + \tag{7}$$

$$+\varepsilon\lambda \int\limits_0^t \big[\|\max\{x(s) : s\in[\theta-h,\theta]\} - x(\theta)\| + \|x(\theta)-\xi(\theta)\|\big]d\theta +$$

$$+\varepsilon\lambda \int\limits_0^t \|\max\{\dot{x}(s) : s\in[\theta-h,\theta]\}\|d\theta \le 2\varepsilon\lambda \int\limits_0^t \|x(\theta)-\xi(\theta)\|d\theta +$$

$$+\varepsilon\lambda[(2A + 2\varepsilon hM + \max(B,\varepsilon M))h + (h+1)\sqrt{n}ML],$$

$$I_2 = \Bigg\| \int\limits_0^t [X(\theta,\xi(\theta),\xi(\theta),0) - \overline{X}(\xi(\theta))]d\theta \Bigg\| \le \tag{8}$$

$$\le 2\lambda ML^2/m + F(\varepsilon,m) \equiv a(\varepsilon,m),$$

where $A = \sup_{t\in[-h,0]} \|\varphi(t)\|$, $B = \sup_{t\in[-h,0]} \|\dot{\varphi}(t)\|$,

$$F(\varepsilon, m) = L\left[\sum_{i=0}^{m-1} \Phi\left(\frac{(i+1)L}{\varepsilon m}, \xi_i\right) + \sum_{i=0}^{m-1} \Phi\left(\frac{iL}{\varepsilon m}, \xi_i\right)\right] +$$

$$+ \max_{0\le k\le m-1} \Phi_0(\varepsilon, \xi_k), \quad \xi_k = \xi\left(\frac{kL}{\varepsilon m}\right), \quad k, m \in \mathbf{N}^+,$$

$$\Phi(t, \xi) = \left\|\frac{1}{t}\int_0^t [x(\theta, \xi, \xi, 0) - \overline{X}(\xi)]d\theta\right\|, \quad \Phi_0(\varepsilon, \xi) = \sup_{0\le\tau\le L} \tau\Phi(\tau/\varepsilon, \xi).$$

Note that for each $\xi \in \Omega(x)$ the function $\Phi(t, \xi)$ tends to zero as $t \to \infty$. Hence choosing m sufficiently large and ε sufficiently small the value of $a(\varepsilon, m)$ can be made arbitrary small [**3**]. Thus it follows from (6)–(8)

$$\|x(t) - \xi(t)\| \le \delta(\varepsilon, m) + 2\varepsilon\lambda \int_0^t \|x(\theta) - \xi(\theta)\|d\theta, \tag{9}$$

whevr

$$\delta(\varepsilon, m) = \varepsilon\lambda[2A + 2\varepsilon hM + \max(B, \varepsilon M))h + (h+1)\sqrt{n}ML] + a(\varepsilon, m),$$

Applying the Gronwall-Bellman lemma to (9) one gets

$$\|x(t) - \xi(t)\| \le \delta(\varepsilon, m)\exp\{2\lambda L\} \text{ for } 0 \le t \le l\varepsilon^{-1}$$

which completes the proof of the theorem.

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