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ON THE $\mathcal{A}$-COMPATIBILITY OF SUPPORTS OF DISTRIBUTIONS OF $\mathcal{K}'\{M_p\}$-TYPE

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Abstract. We determine relations between the notions of $\mathcal{A}$-compatibility and of M_p–convolution of distributions from $\mathcal{K}'\{M_p\}$.

1. Introduction. First we shall repeat two definitions. Let A and B be subsets of $\mathbf{R}$. If these sets satisfy the condition:

$$x_n \in A,\ \ y_n \in B,\ \ |x_n| + |y_n| \to \infty \Rightarrow |x_n + y_n| \to \infty, \quad (n \to \infty) \tag{1}$$

then they are called compatible.

The sets A and B are called polynomially compatible if there exists a polynomial P on $\mathbf{R}$ such that

$$x \in A,\ \ y \in B \Rightarrow |x| + |y| \leq P(|x + y|). \tag{2}$$

It is known that if $f, g \in \mathcal{D}'$ $(f, g \in \mathcal{S}')$ and the sets $A = \operatorname{supp} f$, $B = \operatorname{supp} g$ are compatible (polynomially compatible), then the convolution (tempered convolution) exists. The notion of compatibility of supports of distributions from $\mathcal{D}'$ was investigated for example in [**1**] and the notion of tempered convolution and polynomial compatibility of supports of tempered distributions is introduced and investigated in [**3**], [**4**], [**5**].

In [**5**, Theorems 5.1 and 5.2] Kamiński proved that the notion of compatibility (polynomial compatibility) is essential for the convolution (tempered covolution) of distributions (tempered distributions).

Kamiński's Theorem [**5**]. *Let A and B be subsets of* $\mathbf{R}$ *and let for every two non-negative measures (non-negatire tempered measures) f and g with* $\operatorname{supp} f \subset A$, $\operatorname{supp} g \subset B$, *the convolution (tempered convolution) $f * g$ exist. Then the sets A and B are compatible (polynomially compatible)* .

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The space $\mathcal{S}'$ is an example of a space of $\|'\{M_p\}$-type [**2**]. In [**7**] we generalize the notions of tempered convolution and of polynomial compatibility. We introduced the definition of M_p-convolution of elements from $\|'\{M_p\}$ and the definition of $\mathcal{A}$-compatibility.

In this paper we shall further investigate relations between the notions of $\mathcal{A}$-compatibility and of M_p-convolution. (We use the symbol $\underline{*}$ for this convolution). We shall give conditions on the sequence (M_p) such that the notion of $\mathcal{A}$-compatibility is essential for the M_p-convolution.

2. $\mathcal{A}\{M_p\}$**-compatible sets.** The space of $\mathcal{K}'\{Mp\}$-type, where $\{M_p(x)\}$ is a sequence of continuous functions on $\mathbf{R}$ such that $1 \leq M_1(x) \leq M_2(x) \leq \dots$, was introduced in [**2**] as the dual space of the space $\mathcal{K}\{M_p\}$. The space $\mathcal{K}\{M_p\}$ is a subspace of $\mathbf{C}^\infty(\mathbf{R})$ defined in the following way:

$$\varphi \in \mathcal{K}\{M_p\} \text{ iff } \|\varphi\|_p := \sup\{M_p\}|\varphi^{(q)}(x)| : x \in \mathbf{R}, q \leq p\} < \infty \quad p = 1, 2, \dots$$

Topology in this space is defined by the sequence of norms $(\| \; \|_p; \mathcal{K}\{M\}_p$ is an F-space, and if we suppose:

(N) for every $p \in \mathbf{N}$ there is $p' \in \mathbf{N}$ such that

$$M_p/M_{p'} \in L^1(\mathbf{R}) \text{ and } M_p(x)/M_{p'}(x) \to 0 \text{ as } |x| \to \infty, \qquad ([2])$$

then $\mathcal{K}\{M_p\}$ is an FS space. ($\mathbf{N}$ is the set of natural numbers.)

In this paper we shall suppose that $M_p(x), p \in \mathbf{N}$, are even functions which increase monotonically to infinity when $x \to \infty$. Also, we suppose that the sequence $(M_p(x))$ satisfies:

(N') $\quad (N)$ holds and $M_p(x)/M_{p'}(x) \to 0$ monotonically as $x \to \infty$.

Condition (N') implies that for every $p' \in \mathbf{N}$ which correspond to some $p \in \mathbf{N}$ in (N)

$$M_{p'}(x)/x \to \infty \quad \text{as } x \to \infty. \tag{3}$$

Namely, from the fact that $M_p/M_{p'} \in L^1$ and $M_p/M_{p'}(x) \to 0$ monotonically, it follows that for some $\tilde{x}_{p,p'} > 0$

$$M_p(x)/M_{p'}(x) < 1/x \text{ if } x > \tilde{x}_{p,p'}. \tag{3*}$$

If (3^*) does not hold, then there exists a sequence (x_k) of positive numbers such that $x_{k+1} > 1 + x_k^2$, $k \in \mathbf{N}$, and $M_p(x_k)/M_{p'}(x_k) > x_k^{-1}$. But then

$$\int_{x_1}^{\infty} (M_p(x)/M_{p'}(x))dx \geq \sum_{k=1}^{\infty} x_{k+1}^{-1}(x_{k+1} - x_k) = \infty.$$

Condition (3^*) implies that for a fixed $m \geq 1$

$$M_p(mx) \leq M_{p'}(xm)/(xm) \quad \text{if } x > \tilde{x}_{p,p'}/m.$$

Since $mM_p(x) \le mxM_p(mx)$ if $m \ge 1$ and $x \ge 1$ we obtain that for every $p \in \mathbf{N}$ there exists a $p' \in \mathbf{N}$ and an $x_{p,p'}$ such that

$$\text{(C)} \qquad mM_p(x) \le M_{p'}(mx) \quad \text{if } x > \tilde{x}_{p,p'}.$$

Without loss of generality we can suppose that (3) holds for every $p \in \mathbf{N}$.

For our investigations of M_p-convolutions the following condition on the sequence (M_p) is also needed:

(4) For every $p \in \mathbf{N}$ there is a $p' \in \mathbf{N}$ and a $C_{p,p'} > 0$ such that
$$M_p^2(x) \le C_{p,p'} M_{p'}(x) \text{ for } x > C_{p,p'}.$$

Now we shall give a definition of the set $\mathcal{A}$ that is somewhat different from the definition of this set in [**7**].

We denote by $\mathcal{A}$ a set of non-negative fucntions defined on $\mathbf{R}^+$, bounded on bounded domains, directed according to the ordinary relation $\le$ (i.e. for every f and g from $\mathcal{A}$ there is an $h \in \mathcal{A}$ such $\max\{f(x), g(x)\} \le h(x), x \in \mathbf{R}$) such that:

(A1) If a non-negative function φ defined on $\mathbf{R}^+$ satisfies the inequality $\varphi(x) \le \psi(x), x \in \mathbf{R}^+$ for some $\psi \in \mathcal{A}$, then $\varphi \in \mathcal{A}$;

(A2) There are $\varphi \in A$ and $x_0 \ge 0$ such that $\varphi(x) \ge x$ for $x \ge x_0$;

(A3) For every $\varphi \in A$, $m \in N$ and $n \in N_0$ there is a $\psi \in A$ such that $m\varphi(x+n) \le \psi(x)$, $x \in \mathbf{R}^+$. ($\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.)

Let us suppose that for a given sequence (M_p) and set $\mathcal{A}$ the following condition holds:

(S) For every $p \in \mathbf{N}$ and $\varphi \in \mathcal{A}$ there is a $p' \in \mathbf{N}$ and an $x_{p,p'} > 0$ such that
$$M_p(\varphi(x)) \le M_{p'}(x) \quad \text{if } x > x_{p,p'}.$$

In this case we shall denote the set $\mathcal{A}$ by $\mathcal{A}(M_p)$.

Condition (S) implies some properties of the sequence $(M_p(x))$. For example:

(5) For every $p \in \mathbf{N}$ there are $p' \in \mathbf{N}$ and $x_{p,p'} > 0$ such that
$$M_p(px) \le M_{p'}(x) \text{ if } x > x_{p,p'}.$$

Let us prove this. From (A2) and (A3) it follows that there exists a $\varphi \in \mathcal{A}$ such that $px \le \varphi(x)$, $x \in \mathbf{R}^+$. Therefore (S) implies (5) because $M_p(x)$ is monotonous for $x \ge 0$.

Sequences $(M_p(x))$ which define spaces of exponential distributions quoted in [**7**, part 5], satisfy all the conditions above.

Let (M_p) be a sequence of even, monotonically increasing functions (when $x \to \infty$) for which (N'), (4) and (5) hold. Then, we denote by $\mathcal{B}(M_p)$ the set of all sets $\mathcal{A}(M_p)$.

Proposition 1. $\mathcal{B}(M_p) \neq 0$.

Proof. We denote by $\mathcal{A}$ the set of all non-negative functions which are smaller or equal to some non-negative (on $\mathbf{R}^+$) polynomial of order 1. It is easy to check that $\mathcal{A}_0 \in \mathcal{B}(M_p)$.

We denote by $\mathcal{A}_{\max}(M_p)$ the set defined by $\mathcal{A}_max(M_p) = \bigcup_{\mathcal{A}\in\mathcal{B}} \mathcal{A}(M_p)$. It is easy to prove that $\mathcal{A}_{\max}(M_p) \in \mathcal{B}(M_p)$.

Let $f, g \in \mathcal{K}' M_p$ and $A = \operatorname{supp} f$, $B = \operatorname{supp} g$. As in [**7**] we say that A and B are compatible if there exists a $\varphi \in \mathcal{A}(M_p)$ such that

$$x \in A,\ \ y \in B \Rightarrow |x| = |y| \leq \varphi(|x+y|).$$

We give now Theorem 9 from [**7**] in the following version:

Theorem 2. *If A and B are $\mathcal{A}_{\max}(M_p)$-compatible, then the convolution $f \underline{*} g$ exists. (*$\operatorname{supp} f \in A, \operatorname{supp} \in B$*.)*

Consider now the precise characterization of the sets $\mathcal{A}(M_p)$ for a given sequence $(M_p(x))$ (which satisfies all the conditions mentioned).

Theorem 3. *Let $(M_p(x))$ satisfy the following condition:*

(B) *For every $p \in \mathbf{N}$, $r \in \mathbf{N}$ and $\varepsilon > 0$ there exists $p' \in \mathbf{N}$ and an $x_{p,r,p',\varepsilon} > 0$*

$$\text{such that } M_p^{-1}(M_r(x)) \leq \varepsilon M_p^{-1}(M_{p',\varepsilon}(x)) \text{ if } x > x_{p,r,p',\varepsilon}.$$

Then $\mathcal{A}_{\max}(M_p)$ is the set of all non-negative functions which are smaller or equal to some linear combinations of functions of the form $x \to M_p^{-1}(M_q(x))$, $(p,q) \in \mathbf{N}^2$, $x > 0$, and a constant function.

Proof. We put $\varphi_{p,q}(x) = M_p^{-1}(M_q(x))$, $x \in \mathbf{R}^+$, $(p,q) \in \mathbf{N}^2$ and denote by A the set of all non-negative functions which are smaller or equal to some linear combinations of functions of the form $x \to \varphi_{p,q}(x)$, $(p,q) \in \mathbf{R}^2$, $x > 0$ and a constant function.

We have $0 \leq \varphi_{p,p}(x)$, for every $p \in \mathbf{N}$ and

$$\max\{\varphi_{p_1,q_1}(x), \varphi_{p_2,p_2}(x)\} \leq \varphi_{p_0,q_0}(x), x \in \mathbf{R}^+,$$

where $p_0 = \min\{p_1, p_2\}$, $q_0 = \max\{q_1, q_2\}$. From (5) and (B) it follows that for every $p, q, m \in \mathbf{N}$ and $n \in \mathbf{N}_0$ there exists a $q' \in \mathbf{N}$ and $\tilde{x}$ such that

$$m\varphi_{p,q}(x+n) \leq \varphi_{p,q'}(x) \quad \text{if } x > \tilde{x}.$$

Namely, for sufficiently large $x > 0$ have

$$mM_P^{-1}(M_q(x+n)) \leq mM_p^{-1}(M_q(2x)) \leq mM_p^{-1}(M_{q_1}(x)) \leq M_p^{-1}(M_q,(x)).$$

If for some non-negative function φ on $\mathbf{R}^+$ we have the estimate $\varphi(x) \leq M_p^{-1}(M_q(x))$ if $x > x_{pq} > 0$ for some $(p,q) \in \mathbf{N}^2$, then $\varphi(x) \in A$ because for suitable $C > 0$

$$\varphi(x) \leq M_p^{-1}(M_q(x)) + C, \quad x \in \mathbf{R}^+.$$

Now it is clear that $A \in \mathcal{B}(M_p)$.

Condition (S) implies that if $\varphi \in \mathcal{A}_{\max}(M_p)$, then $\varphi(x) \leq M_p^{-1}(M_p'(x))$ for $x > x_{p,p'} > 0$; that is $\varphi \in A$. Since $A \in \mathcal{B}(M_p)$, the assertion is proved.

Let us remark that the sequences (M_p) given in [**7**, part 5] satisfy condition (B) and that the corresponding sets $\mathcal{A}$ can be redefined to be $\mathcal{A}_{\max}(M_p)$.

3. Conditions for the $\mathcal{A}(M_p)$-compatibility. Let $(M_p(x))$ be a sequence which satisfies all the conditions from Section 2 and let $\mathcal{A}(M_p)$ be an element from $\mathcal{B}(M_p)$ (we suppose that condition (S) is satisfied).

In Theorem 5, which will be stated later, the following condition concerning (M_p) and $\mathcal{A}(M_p)$ will be used:

(B1) For every $p \in \mathbf{N}$ and every $q \in \mathbf{N}$ there exists $\varphi \in \mathcal{A}(M_p)$ and an $x_\varphi > 0$ such that $M_p(\varphi(x)) \geq M_q(x)$ if $x > x_\varphi$.

Proposition 4. (i) *If* $\mathcal{A}(M) = \mathcal{A}_{\max}(M_p)$, *then* (B) *implies* (B1).

(ii) *Conditions* (S) *and* (B1), *concerning the given set* $\mathcal{A} = \mathcal{A}(M_p)$, *imply that* (B) *holds and that* $\mathcal{A}(M_p) = \mathcal{A}_{\max}(M_p)$.

Proof. (i) This follows easily from Theorem 3.

(ii) It follows from (B1) that for every p and every $r \in \mathbf{N}$ there exists a $\varphi \in \mathcal{A}(M_p)$ such that $M_p^{-1}(M_r(x)) \leq \varphi(x)$ for sufficiently large x. From (A3) and (S) it follows that:

$$m\varphi(x) \leq \psi(x), \qquad \text{for some } \psi \in \mathcal{A}(M_p).$$
$$\psi(x) \leq M_p^{-1}(M_{p'}(x)) \text{ for some } p' \text{ and sufficiently large } x.$$

Thus we obtain that for arbitrary $m \in \mathbf{N}$, $p \in \mathbf{N}$, $r \in \mathbf{N}$, there exists a $p' \in N$ and an $\tilde{x} > 0$ such that

$$mM_p^{-1}(M_r(x)) \leq M_p^{-1}(M_{p'}(x)) \text{ if } x > \tilde{x}.$$

that is, condition (B) holds. Since $\mathcal{A}(M_p)$ contains functions of the form $M_p^{-1}\cdot$ $\cdot(M_q(x))$, $(p,q) \in N^2$, and their non-negative linear combinations, it follows that $\mathcal{A}(M_p) = \mathcal{A}_{\max}(M_p)$.

Theorem 5. *We suppose that* $(M_p(x))$ *is a sequence of functions which satisfies all the conditions from Section* 2 *and that* (S) *and* (B1) *hold for a given set* $\mathcal{A}$. *Then the following assertion holds:*

$(*)$ *If A and B are subsets of* $\mathbf{R}$ *such that for every two non-negative measures f and g from $\mathcal{K}'\{M_p\}$ with supports in A and B respectively, the convolution $f * g$ exists, then A and B are $\mathcal{A}_{\max}(M_p)$-compatible.*

Proof. We shall use the idea of the proof of Theorem 5.2. from [**5**]. Since tempered non-negative measures are non-negative measures from $\mathcal{K}'\{M_p\}$, we have that the sets A and B are compatible.

Let us suppose that A and B are not $\mathcal{A}_{\max}\{M_p\}$-compatible.

Let $p \in \mathbf{N}$ be fixed. There are points $x_i \in A$, $y_i \in B$ such that

$$|x_i| + |y_i| \geq 2^i(M_p^{-1}(M_i(|x_i| + |y_i|)) + 1), \quad i \in \mathbf{N}. \tag{6}$$

This holds because functions of the form $2^i(M_p^{-1}(M_i(x) + 1))$. $x \in \mathbf{R}^+$, $i \in \mathbf{N}$, belong to $\mathcal{A}_{\max}(M_p)$. Condition (6) implies that $|x_i| + |y_i| \to \infty$, and therefore, $|z_i| = |x_i + y_i| \to \infty$ as $i \to \infty$.

There are three possibilities:

(i) $|x_i| \to \infty$ and $y_i| \to \infty$;

(ii) $|x_i| \to \infty$ and $|y_i| \not\to \infty$;

(iii) $|x_i| \not\to \infty$ and $y_i| \to \infty$.

First we consider case (i). It is not a restriction if we suppose that $|x_{i+1} > |x_i|$, $|y_{i+1}| > |y_i| + 1$ and $z_{i+1} > |z_i| + 1$, $i \in \mathbf{N}$.

We put

$$f(t) = \sum_{i=1}^{\infty} M_{p'}(|x_i|)\delta(t - x_i); \quad g(t) = \sum_{i=1}^{\infty} M_{p'}(|y_i|)\delta(t - y_i),$$

where we shall choose $p' \in N$ later.

From (5) we obtain that for a given $p \in \mathbf{N}$ there exists a p' and $x_{pp'}$ such that

$$M_p(x) \leq M_{p'}(x/2) \leq M_{p'}(x - t)M_{p'}(t) \quad \text{if } x > x_{pp'} \text{ and } t \in \mathbf{R}. \tag{7}$$

Now we choose p' as an element from $\mathbf{N}$ which corresponds to p (p was fixed earlier) in (7). From (7) we have

$$\begin{aligned} M_{p'}(|x_i|)M_{p'}(|y_i|) &= M_{p'}(|x_i| + |y_i| - |y_i|)M_{p'}(|y_i|) \\ &\geq M_{p'}(|x_i| + |y_i|)/2) \geq M_p(|x_i| + |y_i|) \end{aligned}$$

if $|x_i| > x_{pp'}$ (This is true for all i with $i \geq i_0$ for some i_0.)

Since f and g belong to $\mathcal{K}\{M_p\}$ and $\operatorname{supp} f \subset A$, $\operatorname{supp} g \subset B$, then the convolutions $f \underline{*} g$ and $f * g$ exist and

$$(f \underline{*} g)(t) = (f^* g)(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} M_{p'}(|x_i|)M_{p'}(|y_j|)\delta(t - x_i - y_j).$$

Using (7) and (6) we have

$$\begin{aligned} (f^* g)(t) &\geq \sum_{i=i_0} M_{p'}(|x_i|)M_{p'}(|y_i|)\delta(t - z_i) \geq \sum_{i=i_0} M_p(|x_i| + |y_j|)\delta(t - z_i) \geq \\ &\geq M_p(M_p^{-1}(M_i(|x_i + |y_i|)))\delta(t - z_i) = \sum_{i=i_0} M_i(|z_i|)\delta(t - z_i). \end{aligned}$$

The last series is a distribution which does not belong to $\mathcal{K}'\{M_p\}$. Thus $(f \underline{*} g)(t)$ is not in $\mathcal{K}'\{M\}$ and so this is a contradiction.

Now we consider case (ii). It is not a restriction if we suppose that $|x_{i+1} > |x_i| + 1$, $y_i \to y$ and $|z_{i+1} > |z_i| + 1$, $i \in \mathbf{N}$. From (C) and (5) it follows that there are sequences (p_i) and (L_i) such that $2^i M_i(x) \leq M_{p_i}(x)$ if $x > L_i$.

We choose the sequence (x_i) such that $M_{p_i}(|x_i + y_i|) \geq 2^i M_i(|x_i + y_i|)$, $i \in \mathbf{N}$ (i.e. $|x_i + y_i| > L_i$) and

$$|x_i| + |y_i| \geq M_p^{-1}(M_{p_i}(|x_i + y_i). \tag{8}$$

The existence of the sequence (x_i) for which (8) holds follows from the fact that the functions $M_p^{-1}(M_{p_i}(x))$ are from $\mathcal{A}_{\max}(M_p)$.

Let $f(t) = \sum_{i=1}^{\infty} M_{p'}(|x_i|)\delta(t - x_i)$ and $g(t) = \sum_{i=1} 2^{-i}(M_{p'}(|y_i|)\delta(t - y_i)$. Clearly, f and g are from $\mathcal{K}'\{M_p\}$. Since $f \underline{*} g$ and $f * g$ exist, and

$$f \underline{*} g = f * g = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} M_{p'}(|x_i|)2^{-j} M_{p'}(|y_g|)\delta(t - x_i - y_j) \geq$$

$$\geq \sum_{i=i_0} 2^{-i} M_p(|x_i| + |y_i|I)\delta(t - z_i) \geq \sum_{i=i_0}^{\infty} 2^{-i} M_p(M_p^{-1}(M_{p_i}(|x_i + y_i))) \cdot$$

$$\cdot \delta(t - z_i) \geq \sum_{i=i_0} M_i(|z_i|)\delta(t - z_i),$$

we obtain a contradiction as the last series is not an element from $\mathcal{K}'\{M\}$.

Case (iii) is symmetrical to case (ii), and the proof is complete.

From the preceding Theorem and Proposition 4 we directly obtain:

THEOREM 6. *If in Theorem 5 instead of* (B1) *we suppose that* (B) *holds, and in addition, if we suppose that* $\mathcal{A}(M_p) = \mathcal{A}_{\max}(M_p)$ *(all the otiter conditions are the same as in Theorem 5), then the assertion* (*) *holds.*

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