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# ON THE GENERALIZED GAUSS LEMMA IN LORENTZIAN MANIFOLDS

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**Abstract.** Gauss lemma in Lorentzian manifold has been given in [bf 1] but nothing is known about the generalized notion of the Gauss lemma in the above manifold. The purpose of this paper is to study the generalized Gauss lemma which would be very useful fox the subsequent studies in Lorentzian geometry.

## 1. Introduction

Let $(M, g)$ be a smooth Lorentzian manifold of dimension $n$ with a metric $g$ of signature $(-, +, \dots, +)$. If $TM$ denotes the tangent bundle of $M$ with usual bundle projection $P_M : TM \to M$, then $TP_M : TTM \to TM$ is the tangent linear map. The total space $TTM$ of the tangent bundle to TM can be considered as a vector bundle over $TM$. The Levi-Civita connection $\nabla$ of $(M, g)$ determines a connection mapping $K : TTM \to TM$, (cf. [**2**], [**3**]). It is well known that for every $\nu \in TM$ the fibre $T_\nu TM$ of the canonical projection $P_{TM} : TTM \to TM$ has a unique orthogonal decomposition $T_\nu TM = H_\nu TM \oplus V_\nu TM$, where the subspaces

$$H_\nu TM = \operatorname{Ker} K \restriction T_\nu TM, \qquad V_\nu TM = \operatorname{Ker} TP_M \restriction T_\nu TM,$$

are called the horizontal and vertical subspaces of $T_\nu TM$, respectively. Moreover for any $\nu \in T_x M$, the tangent space of $M$ at $x \in M$; $TP_M \times K$ maps $H_\nu TM \oplus T_x M$, isomorphically onto $T_x M \oplus T_x M$. Furthermore the isomorphisms

$$E_\nu = TP_M \restriction H_\nu TM : H_\nu TM \to T_x M, \text{ and } F_\nu = K \restriction V_\nu TM : V_\nu TM \to T_x M,$$

are isometries for any $\nu \in TM$, where $x = P_M(\nu)$. We define a metric $\overline{g}$ for $TTM$ by using the connection mapping $K$ as follows. For any $X \in T_\nu TM$, $\overline{g}(X, X) = (g(TP_M X), TP_M(X)) + g(K(X), K(X))$. Let $\nu : U \to TM$ be a vector field on a neighborhood $U$ of $x \in M$; then for the covariant derivative of $\nu$ in the direction $w_1 \in T_x M$ we have $w_2 = \nabla_{w_1} \nu = K(T\nu w_1))$, where $T_\nu$ is the tangent linear map defined by $\nu$. Since $T_\nu(w_1) \in T_\nu TM$, then $T_\nu(w_1) = E_\nu^{-1}(w_1) + F_\nu^{-1}(w_2)$ holds

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by the decomposition considered above. More generally, for any $X \in TTM$, $X = E_\nu^{-1}(w_1) + F_\nu^{-1}(w_2)$ holds, where $w+1 = TP_M(X)$, $w_2 = K(X)$ and $\nu = P_{TM}(X)$.

Here we note that any $X \in TTM(P_{TM}(X) \neq 0)$ defines uniquely a Jacobi field $Y : [0,1] \to TM$ along time like geodesic $c : [0,1] \to M$ such that $c(0) = \nu = P_{TM}(X)$ with the following initial conditions

$$Y(0) = w_1 = TP_M(X), \qquad \nabla Y(0) = w_2 = K(X).$$

## 2. Space like submanifold in Lorentzian manifold

Let $(M, g)$ be a smooth Lorentzian manifold. The smooth submanifold $L$ of dimension $m$ of $M$ is said to be nondegenerate if for each $x \in L$ and nonzero vector $\nu \in T_xL$, the tangent space of $L$ at $x$, there exists some $w \in T_xL$ with $g(\nu, w) \neq 0$. lf in addition, $g \restriction T_xL \times T_xL$ is positive definite for each $x \in L$, then $L$ is said to be a space like submanifold [**1**].

For each $x \in L$, there is a well defined subspace $N_xL$ of tangent space $T_xM$ such that $N_xL = \{\nu \in T_xM |\, g(\nu, w) = O$ for all $w \in T_xL\}$ with the property $T_xL \cap N_xL = \{0\}$. This subspace is called the normal space to $L$ at $x$. Let $N(L) = \bigcup\{N_xL |\, x \in M\}$ be the set of all normal vectors to $M$. Then $N(L)$ is a subbundle of $TM$, called the normal bundle of $L$. Geometrically the normal bundle $N(L)$ bears a strong analogy to the tangent bundle $TM$. Just as vector fields in $\mathcal{H}(M)$ are the sections of $TM$, those in $\mathcal{H}^\perp(L)$ are the sections of $N(L)$. Here $\mathcal{H}(M)$ is the set of all smooth vector fields on $M$.

Let $\nu : U \to N(L)$ be a section of the normal bundle $N(L)$ on a neighborhood $U$ of $x \in L$, then for the covariant derivative of $\nu$ in the direction $w_1 \in T_xL$ we have $w_2 = \nabla_{w_1}\nu = K(T\nu(w_1))$ holds as in the case of tangent bundle $TM$. Since the tangent space $T_xM$ has an orthogonal decomposition $T_xM = T_xL \oplus N_xL$ it follows that the decomposition $\nabla_{w_1}\nu = A_\nu(w_1) + \tilde{w}_2$ is valid, where $A_\nu : T_xL \to T_xL$ is the shape operator derived from $\nu \in N_xL$ and $\tilde{w}_2 = \tilde{\nabla}_{w_1}\nu \in N_xL$.

Now the inclusions $N(L) \subset TM$ and $TN(L) \subset TTM$ permit us to consider the induced connection map $\tilde{K} : TN(L) \to N(L)$ such that $\tilde{K}(X) = K(X) - A_\nu(w_1)$ for any $X \in TN(L)$ where $P_{TM} \restriction N(L)$, $\nu = P_{TM}(X) \in N_xL$ and $TP_M \restriction N(L)$, $w_1 = TP_M(X) \in T_xL$. Thus the induced covariant derivative of $\nu$ in the direction $w_1$ is given by $\tilde{w}_2 = \tilde{\nabla}_{w_1}\nu = K(T\nu(w_1)) \in N_xL$. Thus the canonical metric $\tilde{g}$ of the normal bundle $N(L)$ is given by

$$\tilde{g}(X, X) = g(TP_M(X), TP_M(X)) + g(K(X), K(X))$$

for any $X \in TN(L)$. As in the case of $TM$ we see that for every $\nu \in N(L)$ the fibre $T_\nu N(L)$ of the canonical projection $P_{TM} \restriction TN(L) \to N(L)$ has a unique orthogonal decomposition $T_\nu N(L) = \tilde{H}_\nu N(L) \oplus \tilde{V}_\nu N(L)$, were the subspaces

$$\tilde{H}_\nu N(L) = \operatorname{Ker} \tilde{K} \restriction T_\nu N(L), \qquad \tilde{V}_\nu N(L) = \operatorname{Ker} TP_M \restriction T_\nu N(L),$$

are called the horizontal and the vertical subspaces of $T_\nu N(L)$ respectively. Further, the isomorphisms

$$\tilde{E}_\nu = TP_M \restriction \tilde{H}_\nu N(L) : \tilde{H}_\nu N(L) \to T_x L, \text{ and}$$
$$\tilde{F}_\nu = \tilde{K} \restriction \tilde{V}_\nu N(L) : \tilde{V}_\nu N(L) \to N_x(L),$$

are isometries for any $\nu \in N(L)$, where $P_M(\nu) = x \in L$. Here we note that for any $X \in TN(L) \subset TTM$ we already have two decompositions. In the first case we have orthogonal decomposition $X = X_{\text{hor}} + X_{|}$, where

$$X_{\text{hor}} = E_\nu^{-1}(w_1), \quad X_{|} = F_\nu^{-1}(w_2),$$

with the metric $\overline{g}$ of $TM$. In the second case $X$ has decomposition

$$X = \tilde{X}_{\text{hor}} + \tilde{X}_{|},$$

where $\tilde{X}_{\text{hor}} = X_{\text{hor}} + F_\nu^{-1}(A_\nu(w_1))$, $\tilde{X}_{|} = X_{|} - F_\nu^{-1}(A_\nu(w_1))$, and this decomposition is evidently orthogonal with respect to the metric $g$ of the normal bundle $N(L)$.

Now we give the characterization of the $L$-Jacobi field. Let $\nu \in N(L)$ be a non-zero vector and $\tilde{c}(t) = t \cdot \nu$ be a radial vector of $N(L)$ then, $\dot{\tilde{c}}(1) \in T_\nu N(L)$. Let $X \in TN(L)$ be such that $\nu = P_{TM}(X) \neq 0$ and $g(\dot{\tilde{c}}(1), X) = 0$ hold. Then X defines uniquely an $L$-Jacobi field $Y$ along the geodesic $c(t) = \exp^\perp L(t \cdot \nu)$, such that $\dot{c}(0) = \nu = P_{TM}(X) \in N_x L$, with the initial conditions

$$Y(0) = w_1 = TP_M(X) \in T_x L, \qquad \nabla Y(0) = w_2 = K(X)$$

where $\exp^\perp : N(L) \to M$ is a normal exponential map. Here we note that $g(\nu, w_1) = 0$ and $g(\nu, w_2) = g(\dot{\tilde{c}}(1), X) = 0$ imply thet $g(\dot{c}, Y) = 0$ is valid for every parameter along the geodesic $c(t)$. Further, $\tilde{K}(X) = K(X) - A_\nu(w_1) = w_2 = A_\nu(w_1) \in N_x L$ holds.

## 3. Generalized Gauss lemma

THEOREM. *Let $L$ be a space like submanifold of a lorentzian manifold $(M, g)$ and let $\exp^\perp : N(L) \to M$ be a normal exponential map. Let $\nu \in N(L)$ be a nonzero vector in the domain of the normal exponential mapping and $\tilde{c}(t) = t \cdot \nu$ be a radial vector of $N(L)$. Then for any $X \in T_\nu N_x L$*

$$\tilde{q}(\dot{\tilde{c}}(1), X) = g(T_\nu \exp^\perp(\dot{\tilde{c}}(1), T_\nu \exp^\perp(X))$$

*holds, where $T_\nu \exp^\perp$ denotes the tangent linear map of $\exp^\perp$ at $\nu$.*

*Proof.* Let $c(t) = \exp^\perp(t \cdot \nu)$ be a time like geodesic. Let $\alpha(t, s) = \exp^\perp(t \cdot \nu(s))$ for $t, s \in R$ be the variation of the geodesic $c(t)$. Then $X \in T_\nu N(L)$ such that $X = d\nu(s)/ds|_{s=0}$ defines a Jacobi field $Y$ along the geodesic $c(t)$ with the following initial conditions

$$\dot{c}(0) = \nu = P_{TM}(X), \qquad Y(0) = w_1 = TP_M(X), \qquad \nabla Y(0) = w_2 = K(X).$$

Also $Y(1) = T_\nu \exp^\perp(X)$, and $\dot{c}(1) = T_\nu \exp^\perp(\dot{\tilde{c}}(1))$, are valid. Now the Jacobi equation [**1**] $\nabla^2 Y + R(Y, \dot{c})\dot{c} = 0$, where $R(Y, \dot{c})\dot{c}$ is the curvature tensor of $M$, implies $g(\nabla^2 Y, \dot{c}) = 0$. This gives

$$\begin{aligned} &g(\nabla Y, \dot{c}) = A, \\ &g(Y, \dot{c}) = At + B, \quad A, B \text{ are constants.} \end{aligned} \tag{3.1}$$

Using the relation $\tilde{g}(X, X) = g(TP_M X, TP_M X) + g(\breve{K}(X), \tilde{K}(X))$, and values of $X$, $TP_M X$, $K(X)$ we find that

$$g(\dot{\tilde{c}}(1), X) = g(0, w_1) + g(\nu, w_2 - A_\nu(w_1)) = g(\dot{c}(0), \nabla Y(0)),$$

as $g(\nu, A_\nu(w_1)) = 0$.

Applying the initial conditions we find

$$B = g(Y(O), \dot{c}(O)) = 0, \qquad A = g(\nabla Y(O), \dot{c}(O)) = \tilde{g}(\dot{\tilde{c}}(1), X).$$

Taking $t = 1$ in (3.2) and using the values of $A$ and $B$ we have

$$g(\dot{c}(1), Y(1)) = \tilde{g}(\dot{\tilde{c}}(1), X),$$

which is the desired result.

It may be noted that the generalized Gauss lemma reduces to the Gauss lemma in the case when the space like submanifold $L$ is a single point [**1**].

COROLLARY. *Let $L$ be a space like submanifold of the Lorentzian manifold $(M, g)$. Let $\exp^\perp L : N(L) \to M$ be the normal exponential map. If $\nu \in N(L)$ is a nonzero vector of $N(L)$, then the kernel of the tangent linear map $T_\nu \exp^\perp$ is orthogonal to $\dot{\tilde{c}}(1)$ where $\dot{\tilde{c}}$ is the radial vector defined by $\nu$.*

*Proof.* Let $X \in \operatorname{Ker} T_\nu \exp^\perp$, then $g(T_\nu \exp^\perp(\dot{\tilde{c}}(1)), T_\nu \exp^\perp(X)) = 0$. Using generalized Gauss lemma we have $g(\dot{\tilde{c}}(1), X) = 0$. This gives the required result.

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