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A PROPERTY BETWEEN COMPACT AND STRONGLY COUNTABLY COMPACT

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Abstract. In this paper we consider a class of spaces called hypercountably compact (hcc)spaces. The class of countably compact and the class of strongly countably compact (scc) spaces contain the class of hypercountably compact spaces. In example 2.1, we give a strongly countably compact space which is not hypercountably compact. In the class of spaces satisfying the first axiom of countability the notions hcc and scc coincide (Theorem 2.3). Some equivalent conditions for a space to be hcc are given in Theorem 2.2. The hcc property is not a continuous invariant (Example 2.4). In section 3 we consider compact spaces which contain noncompact hcc (scc) spaces as subspaces. In section 4 we also consider strongly sequentially compact (ssc) spaces.

1. Introduction

The closure of a subset A of a space X is denoted by $\mathrm{cl}_X(A)$. In this paper we assume that all spaces are Hausdorff (T_2-spaces). For notations and definitions not given here see [**1**], [**3**], [**5**].

Definition 1.1. [**4**]. A space X is strongly countably compact (scc) if every countable subset in X has a compact closure in X.

Definition 1.2. A space X is hypercountably compact (hcc) if the union of every countable family of compact sets in X has a compact closure in X.

A topological space X is called countably compact if every countable open cover of X has a finite subcover.

The following diagram illustrates the implications among the compactness properties that we consider:

$$\text{compactness} \to \begin{array}{c}\text{hypercountably}\\ \text{compactness}\end{array} \to \begin{array}{c}\text{strongly countably}\\ \text{compactness}\end{array} \to \begin{array}{c}\text{countably}\\ \text{compactness}\end{array}$$

Counterexamples show that the implications in the diagram are not reversible. That countable compactness does not imply strongly countable compactness, see

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[**3**]. In example 2.1 the space is strongly countably compact but it is not hypercountably compact. Let $[0,\omega_1)$ be the space of ordinals less than the first uncountable ordinal with the order topology. The space $[0,\omega_1)$ is hypercountably compact but is not compact: Let $\mathcal{K} = \{K_n : n \in N\}$ be a countable family of compact sets in $[0,\omega_1)$. Let $a_n = \sup(K_n)$, $n \in N$. Since the least upper bound of any countable subset of $[0,\omega_1)$ is countable and will be strictly less than ω_1, there exists $a \in [0,\omega_1)$, $a = \sup(\{a_1 \in [0,\omega_1) : n \in N\})$ and is strictly less than ω_1. The set $K = [0,a]$ is compact in $[0,\omega_1)$ and $K_n \subset K$ for all $n \in N$. Hence, by definition 1.2, the space $[0,\omega_1)$ is hypercountably compact. The space $[0,\omega_1)$ fails to be compact since the collection $\{[0,x) : x > \omega_1\}$ is an open cover with no finite subcover.

Let X be a topological T_2-space. Then:

(1) $\exp(X)$ denotes the space of all non-empty closed subsets of X with finite topology. The finite topology on $\exp(X)$ is the one generated by open collection on the form $\langle U_1, U_2, \dots, U_n\rangle = \{F \in \exp(X) : F \subset \bigcup_{i=1}^n U_i$ and $F \cap U_i \neq \emptyset$, for $i = 1,2,\dots,n\}$ where $U_1, U_2, \dots, U_n$ are open subsets of X. The finite topology on $\exp(X)$ is also known as the exponential topology or the Vietoris topology. The space $\exp(X)$ is known as the hyperspace of X (see [**9**], [**10**], [**11**]).

(2) $\mathcal{K}(X)$ denote the set of all non-empty compact subsets of X as a subspace of $\exp(X)$.

(3) $\mathcal{K}^{(2)}(X) = \mathcal{K}(\mathcal{K}(X))$, $\mathcal{K}^{(n)}(X) = \mathcal{K}(\mathcal{K}^{(n-1)}(X)), n = 2,3,\dots$

(4) $\mathcal{F}_n(X) = \{F \subset X : F$ has at most n points$\} \subset \mathcal{K}(X)$.

(5) $\mathcal{F}(X) = \{F \subset X : F$ is finite$\} \subset \mathcal{K}(X)$.

Definition 1.3. [**1**] A quasi-ordered and directed set $(S,\leq)$ is $\aleph_0$-directed if for every countable subset $S_0 = \{s_1, s_2, \dots, s_n, \dots\} \subseteq S$ there exists an $s \in S$ such that $s_n \leq s$, for all $n \in N$.

2. Relationship between hcc and scc spaces and some properties of hcc spaces

It is clear that every hcc space is a scc space. The following example shows that not every strongly countably compact space is hypercountably compact.

Example 2.1. (A strongly countably compact space which is not hypercountably compact). Let $[0,\omega_1]$ ($[0,\omega_0]$) be the the space of ordinals less than or equal to the first uncountable ordinal (first countable ordinal) with the order topology. Let $X_1 = [0,\omega_1] \times [0,\omega_0]$. Since ordinal space $[0,a]$ (a is any ordinal) is compact and Hausdorff, so is the Tychonoff product X_1. Since every compact Hausdorff space is normal, X_1 is normal. Let $X_2 = [0,\omega_1] \times [0,\omega_0] - \{(\omega_1, n) : n \in [0,\omega_0)\}$ be subspace of X_1. Then X_2 is noncompact and normal in the subspace topology. Furthermore, X_2 is hypercountably compact: Let $\mathcal{D} = \{K_n \in \mathcal{K}(X_2) : n \in N\}$ be a countable family of compact sets in X_2 and let $(\omega_1,\omega_0) \notin K_n$ for all $n \in N$. The sequences $\mathcal{D}' = \{K_n' \in \mathcal{K}([0,\omega_1)) : n \in N\}$ and $\mathcal{D}'' = \{K_n'' \in \mathcal{K}([0,\omega_1) : n \in N\}$ are projections of $\mathcal{D}$ onto $[0,\omega_1]$ and $[0,\omega_0)$. Sets K_n' and K_n'' are compact and

$K_n' \subseteq [0,\omega_1)$, $K_n'' \subseteq [0,\omega_0)$ for all $n \in N$. Let $a_n = \sup(K_n')$, $n \in N$. Since the least upper bound of any countable subset of $[0,\omega_1)$ is countable and is strictly less than ω_1 there exists $a \in [0,\omega_1)$, $a = \sup(\{a_n \in [0,\omega_1) : n \in N\})$ and strictly les than ω_1. The set $K = [0,a) \times [0,\omega_0]$ is compact in X_2 and $K_n \subseteq K$ for al $n \in N$. If $(\omega_1,\omega_2) \in K_n$ for infinite number of members $K_n \in \mathcal{D}$, then the set $K' = [0,a] \times [0,\omega_0] \cup ([0,\omega_1] \times \{\omega_0\})$ is compact in X_2 and $K_n \subseteq K'$ for all $n \in N$. Hence X_2 is a hypercountably compact space. Let $X_3 = X_2 \cup \{p\}$, $(p \notin X_2)$ be the one-point compactification of X_2. So we get the followig diagram

$$\begin{array}{cccc}
(0,\omega_0) & \ldots\ldots\ldots\ldots\ldots\ldots & (\omega_1,\omega_0) & \\
 & \ldots\ldots\ldots\ldots\ldots\ldots & & \\
 & \ldots\ldots\ldots\ldots\ldots\ldots & & \cdot\, p \\
 & \ldots\ldots\ldots\ldots\ldots\ldots & & \\
(0,0) & \ldots\ldots\ldots\ldots\ldots\ldots & (\omega_1,0) &
\end{array}$$

which we will call the space X_3. Then:

(1) X_3 is compact and a T_1 space.

(2) X_3 is not Hausdorff since the point p and (ω_1,ω_0) have no disjoint neighborhoods.

(3) The point (ω_1,ω_0) is an accumulation (limit) point of X_3.

(4) Let $X_4 = X_3 = \{(\omega_1,\omega_0)\}$ be a subspace of X_3. The set $A = \{(a,\omega_0) : 0 \le a < \omega_1\}$ is a subset of X_4. Furthermore, the set A is a closed subset of X_4 homeomorphic to $[0,\omega_1)$. Hence X_4 is not compact since the space $[0,\omega_1)$ is not compact.

(5) The space X_4 is Hausdorff but it is not regular since the point p and the set $A = \{(a,\omega_0) : 0 \le a < \omega_1\}$ have no disjoint neighborhoods.

(6) Let $A = \{a_n \in X : n \in N\}$ be any countable subset of X_4 and let $p \in A$. Then $A = \{(x_n,y_n) : x_n \in [0,\omega_1), y_n \in [0,\omega_0]; n \in N\}$ where $\{x_n \in [0,\omega_1) : n \in N\} \subset [0,\omega_1)$ and $\{y_n \in [0,\omega_0] : n \in N\} \subseteq [0,\omega_0]$. Let a be an upper bound for the x_n; $a < \omega_1$ since ω_1 has uncountably many predecessors, while a has only countably many. Thus the set $[0,a] \times [0,\omega_0]$ is closed and compact in X_4. Furthermore, $A \subseteq [0,a] \times [0,\omega_0)$. Then $\mathrm{cl}_{X_4}(A) \subseteq [0,a] \times [0,\omega_0]$ and $\mathrm{cl}_{X_4}(A)$ is a compact subset of X_4. Hence X_4 is a strongly countably compast space.

(7) The space X_4 is not hypercountably compact because there exists a countable family $\mathcal{F} = \{([0,\omega_1) \times \{k\}) \cup \{p\} : k = 0,1,2,\ldots\}$ of non-empty compact subsets of family X_4 such that:

$\mathrm{cl}_{X_4}(|\mathcal{F}|) = \mathrm{cl}_{X_4}(\bigcup\{([0,\omega_1) \times \{k\}) \cup \{p\} : k = 0,1,\ldots\}) = X_4$. Hence the space X_4 is scc but not hcc.

The following result gives a characterization of hcc spaces.

Theorem 2.2. *Let X be a T_2-space. The following are equivalent:*

(1) *X is hypercounlably compact,*

(2) *$\mathcal{K}(X)$ is $\aleph_0$-directed by inclusion,*

(3) $\mathcal{K}(X)$ *is countably compact,*

(4) $\mathcal{K}(X)$ *is strongly countably compact,*

(5) $\mathcal{K}(X)$ *is hypercountably compact,*

(6) $\mathcal{K}^{(n)}(X)$ *is hypercountably cornpact.*

Proof. It is clear that $(1) \leftrightarrow (2)$. Let $\{\mathcal{K}_n \in \mathcal{K}(\mathcal{K}(X)) : n \in N\}$ be a countable table family of compact sets in $\mathcal{K}(X)$. Then $|\mathcal{K}_n| = \bigcup\{K \in \mathcal{K}_n\} \in \mathcal{K}(X')$ for all $n \in N$. Since X is a hcc space then $\mathrm{cl}_X(\bigcup\{|\mathcal{K}_n| : n \in N\}) \in \mathcal{K}(X)$ which implies that $\exp(\mathrm{cl}_X(\bigcup\{|\mathcal{K}_n| : n \in N\}))$ is a compact subset of $\mathcal{K}(X)$ and $\mathcal{K}_n \subseteq \exp(\mathrm{cl}_X(\bigcup\{|\mathcal{K}_n| : n \in N\}))$ for all $n \in N$. Hence $\mathcal{K}(X)$ is a hcc space and $(1) \leftrightarrow (5)$.

$(4) \rightarrow (1)$: Suppose that $\mathcal{K}(X)$ is a scc space and let $\mathcal{K}\{K_n \in \mathcal{K}(X) : n \in N\}$ be any countable family of compact subsets of X. Then $\mathrm{cl}_{|CalK(X)}(\mathcal{K})$ is a compact subspace of $\mathcal{K}(X)$. Since $\mathrm{cl}_{|CalK(X)}(\mathcal{K})$ is compact we have (see [**11**]) $|\mathrm{cl}_{|CalK(X)}(\mathcal{K})| = \bigcup\{K \in \mathrm{cl}_{|CalK(X)}(\mathcal{K})\}$ is a compact subset of X and $K_n \subseteq |\mathrm{cl}_{|CalK(X)}(\mathcal{K})| = \bigcup\{K \in \mathrm{cl}_{|CalK(X)}(\mathcal{K})\}$ for all $n \in N$. Hence X is a hcc space.

$(1) \rightarrow (4)$: Since $(1) \rightarrow (5)$ and $(5) \rightarrow (4)$ we have $(1) \rightarrow (4)$.

$(5) \rightarrow (1)$: Since $(5) \rightarrow (4)$ and $(4) \rightarrow (1)$ we have $(5) \rightarrow (1)$.

$(5) \leftrightarrow (6)$: Can be obtained in similar way.

$(5) \rightarrow (3)$: Since $(5) \rightarrow (4)$ and $(4) \rightarrow (3)$ we have $(5) \rightarrow (3)$.

$(3) \rightarrow (5)$: Let $\mathcal{K}(X)$ be countably compact and let $\mathcal{K} = \{K_n \in \mathcal{K}(X) : n \in N\}$ be a countable family of $\mathcal{K}(X)$ increasing by inclusion. Since $\mathcal{K}(X)$ is countably compact there exists a $K \in \mathcal{K}(X)$ such that K is an accumulation point of $\mathcal{K}$. Let $F = \mathrm{cl}_X(\bigcup\{K_n \in \mathcal{K}\}) = \mathrm{cl}_X(\bigcup\{K_n \in \mathcal{K}(X) : n \in N\})$ and let $\langle U_1, U2, \ldots, U_n\rangle$ be a basic open set containing F. Then $F \subseteq \bigcup_{i=1}^{m} \bigcup_i$ and $F \cap U_i \neq \emptyset$ for $i \in \{1, 2, \ldots, m\}$. Then for each $n \in N$ we have that $K_n \subseteq \bigcup\{U_i : i \in \{1, 2, \ldots, m\}\}$ and for all $i \in \{1, \ldots, m\}$ there exists an $n_i \in N$ such that $U_i \cap K_{n_i} \neq \emptyset$. Let $n_0 = \max(\{n_i : i \in \{1, 2, \ldots, m\}\})$. Then for each $n > n_0$ we have $K_n \cap U_i \neq \emptyset$, $i \in \{1, 2, \ldots, m\}$ and it follows that $K_n \in \langle U_1, U_2, \ldots, U_m\rangle$. Hence F is an accumulation point of $\mathcal{K}$. Since $\mathcal{K}(X)$ is countably compact, $F \in \mathcal{K}(X)$. Furthermore; for each $K \in \mathcal{K}(X)$, $K \neq F$, we have that K is not an accumulation point of $\mathcal{K}$: Let $K \neq F$. Then there is a point $x \in K$ and $x \notin F$ of there is a point $x \in F$ and $x \notin K$.

Case I: $x \in K$ and $x \notin F$. Since $F \subset X$ is compact and X in Hausdorff, there exists an open set U_x containing x such that $U_x \cap F = \emptyset$. Then $F \notin \rangle U_x \langle^*$ and $K \in \rangle U_x \langle$. Furthermore, for each $n \in N$, we have $K_n \notin \rangle U_x \langle$ and $K \in > U_x <$. Hence K is not an accumulation point of $\mathcal{K}$.

Case II: $x \in F$ and $x \notin K$. Since X is Hausdorff and K, F are compact subsets of X, there exist open neighborhoods U of K and U of x such that $U \cap U_x = \emptyset$. Then $K \in \langle U\rangle$ and $F \notin \langle U\rangle$ which implies that $K_n \notin \langle U\rangle$ for all $n \in$N. Hence K is not an accumulation point of $\mathcal{K} = \{K_n \in \mathcal{K}(X) : n \in N\}$. Since $F = \mathrm{cl}_X(\bigcup\{K_n \in$

$^*\rangle U\langle$ stands for $\{F \in \exp(X) : F \cap U \neq \emptyset\}$.

$\mathcal{K}\}) = \mathrm{cl}_X(\bigcup\{K_n \in \mathcal{K}(X) : n \in N\})$ is contained in $\mathcal{K}(X)$ by 2.1, the space X is hypercountably compact.

Suppose that $\mathcal{K} = \{K_n \in \mathcal{K}(X) : n \in N\}$ is not increasing by inclusion. Then the family $\mathcal{K}' = \{K'_n \in \mathcal{K}(X) : n \in N\}$ with $K'_n = \bigcup_{i=1}^{n} K_i$ is increasing by inclusion and, by the foregoing, there exists a compact set F' of X such that $F' = \mathrm{cl}_X(\bigcup\{K' \in \mathcal{K}'\})$. Since for each $n \in N$ we have $K_n \subseteq K'_n$, $F = \mathrm{cl}_X(\bigcup\{K_n \in \mathcal{K}\})$ is a compact subset of X. Hence X is hypercountably compact. This completes the proof.

THEOREM 2.3. *Let X be a first countable strongly countably compact T_2-space. The X is a hypercountably compact space.*

Proof. To prove 2.3, it suffices by 2.2, to show that $\mathcal{K}(X)$ is a scc space. Since X is a first countable T_2-space, then $\mathcal{K}(x)$ is also a first countable T_2 space (see [**11**]). Let $\mathcal{F}$ denote the family of all finite subsets of X. Then $\mathcal{F} \subset \mathcal{K}(X)$ and $\mathrm{cl}_{\mathcal{K}(X)}(\mathcal{F}) = \mathcal{K}(X)$ (see [**9**], [**11**]). Let $\mathcal{K}$ be any countable family of compact sets in X, $\mathcal{K} = \{K_n \in \mathcal{K}(X) : n \in N\}$. Since $\mathcal{K}(X)$ is a first countable space and $\mathrm{cl}_{\mathcal{K}(X)}(\mathcal{F}) = \mathcal{K}(X)$, then for each $K_n \in \mathcal{K}$ there exists $\mathcal{A}^{(n)} = \{A^n_k : k \in N\}$ such that $\hat{K} = \mathrm{cl}_{\mathcal{K}(X)}(\mathcal{A}^{(n)})$ ($\hat{K} = \{\{x\} : x \in K_n\}$ is a compact subset of $\mathcal{K}(X)$ homeomorphic to K_n) for all $n \in N$. The family $\mathcal{A} = \{A^n_k : (k,n) \in N \times N\}$ is also a countable family and $|\mathcal{A}| = \bigcup\{A^n_k : (k,n) \in N \times N\}$ is also a countable subset in X. Since X is a strongly countably compact space, then $K = \mathrm{cl}_X(|\mathcal{A}|)$ is a compact subset of X and $\exp(K)$ is also a compact subset of $\mathcal{K}(X)$. Now we have $K \in \exp(K) \subseteq \mathcal{K}(X)$, and $\exp(K)$ is a compact subset od $\mathcal{K}(X)$ which implies that $\mathrm{cl}_{\mathcal{K}(X)}(\mathcal{K}) \subseteq \exp(K)$. Hence $\mathrm{cl}_{\mathcal{K}(X)}(\mathcal{K})$ is a compact subset of $\mathcal{K}(X)$ and $\mathcal{K}(X)$ is an scc space. This completes the proof.

It can be shown that every continuous image of an scc space is an scc space. The following example shows that the continuous image of a hcc need not be an hcc space.

Example 2.4. Let $X = [0,\omega_1] \times [0,\omega_0] - \{(\omega_1, x) : x \in [1,\omega_0)\}$ be the subspace of the product $[0,\omega_1] \times [0,\omega_0]$. Then, by example 2.1, X is normal and hypercountably compact in the subspace topology. Let $Y = X_4$ where X_4 is the space in example 2.1. The mapping $f : X \to Y$ is defined by $f(x) = x$, for all $x \neq (\omega_1, 0)$ and $f(\omega_1, 0) = p$ is continuous surjection. The space X is hcc, but Y is not a hcc space.

Remark. A continuous surjection $f : X \to Y$ is a compact-covering if whenever B is a compact set in Y, there exits a compact set A in X such that $(A) = B$. If $f : X \to Y$ is a compact-covering mapping and X is a hcc space, then Y is hcc.

We now state, without proof, some properties of hcc spaces.

PROPOSITON 2.5. *A closed subset A of a hcc space X is itself hcc.*

PROPOSITION 2.6. *Each hcc subset of a first countable T_2-space X is closed in X.*

Example 2.7. Let $X = [0, \omega_1]$ and $A = [0, \omega_1) \subset X$. The subset A is hcc but A is not closed in X. The space $X = [0, \omega_1]$ is a T_2-space but is not first countable.

PROPOSITION 2.8. *Let* $(X_a \cdot a \in A)$ *be a family of non-empty spaces. Then the product space* $X = \prod\{x_a : a \in A\}$ *is hcc if and only if* X_a *is hcc for each* $a \in A$,

3. P-points, weak P-points and some characterizations

Definition 3.1. Let X be a topological T_2-space.

(1) A point $p \in X$ is said to be a P-point provided that each intersection of countably many neighbourhoods od p is a neighbourhood of p.

(2) A point $p \in X$ is a weak P-point if $p \notin \mathrm{cl}_X(F)$ for each countable $F \subset X - \{p\}$.

It is easy to see that every P-point is a weak P-point. In example 2.1, the point (ω_1, ω_0) is a weak P-point but is not a P-point.

THEOREM 3.2. *Let* X *be a compact* T_2*-space and* $p \in X$ *be a limit point in* X*. Then:*

(1) *The subspace* $X - \{p\}$ *is an scc space if and only if the point* p *is a weak* P*-point.*

(2) *The subspace* $X - \{p\}$ *is a hcc space if and only if the point* p *is a* P*-po in t.*

Proof. $X - \{p\}$ is an open noncompact subset of X.

(1) Suppose that $X - \{p\}$ is an scc space and let $A \subset X - \{p\}$ be a countable set. Then $\mathrm{cl}_{X-\{p\}}(A) = \mathrm{cl}_X(A)$ is a compact set in $X - \{p\}$ and $p \notin \mathrm{cl}_X(A)$. Hence point p is a weak P-point in X.

Conversely, suppose that p is a weak P-point in X. Let $A \subset X\{p\}$ be a countable set. Then $p \notin \mathrm{cl}_X(A)$ which implies that $\mathrm{cl}_X(A) = \mathrm{cl}_{X-\{p\}}(A)$. Since $\mathrm{cl}_X(A)$ is a compact (the space X is compact and Hausdorff) subset in X, then $\mathrm{cl}_{X-\{p\}}(A)$ is a compact subset in $X - \{p\}$. Hence, subspace $X - \{p\}$ is an space.

(2) Suppose that $X - \{p\}$ is an hcc space and let $\{U_n(p) : n \in N\}$ be any countable family of open neighbourhoods of the point p in X. Then $\bigcap\{U_n(p) : n \in N\} = \bigcup\{X - U_n(p) : n \in N\}$ and $X - U_n(p) \subset X - \{p\} \subset X$ is a compact set for all $n \in N$. Since $X - \{p\}$ is a hcc space there exists a compact set $K \subset X - \{p\}$ such that $X - U_n(p) \subset K$ for all $n \in N$. The set $U = X - K$ is an open neighbourhood of p in X. Therefore, $X - K \subset \bigcap\{U_n(p) : n \in N\}$. Hence p is a P-point in X.

Conversely, suppose that p is a P-point in X and let $\{K_n : n \in N\}$ be any countable family of compact sets in $X - \{p\}$. Then $\bigcup \{K_n : n \in N\} = \bigcap\{X - K_n : n \in N\}$ and $X - K_n$ is an open neighbourhood of p in X for all $n \in N$. Since p is a P-point in X, there exists an open neighbourhood U of p such that $U \subseteq \bigcap\{X - K_n : n \in N\}$. Therefore, $K_n \subseteq X - U$ for each $n \in N$ and $X - U$ is a compact subset of $X - \{p\}$. Hence, by definition 1.2, subspace $X - \{p\}$ is an hcc space. This completes the proof.

COROLLARY 3.3. *Let X be a compact T_2-space and $p \in X$ be a limit point in X. If P is a weak P-point which is not a P-point, then the subspace $X - \{p\}$ is a strongly countably compact space which is not a hypercountably compact space.*

Example 3.4. Let βN denote the Stone-Čech compactification of positive integers N and $N^* = \beta N - N$. Kunen in [**7**] has shown using Martin's Axiom that there exists a weak P-point p in N^* which is not a P-point. By corollary 3.3, the space $X = N^* - \{p\}$ is an example of strongly conntably compact space which is not hypercountably compact space.

4. Strongly sequentially compact (ssr) spaces

Definition 4.1. A space X is strongly sequentially compact (ssc) if $\mathcal{K}(X)$ is sequentially compact.

PROPOSITION 4.2. *Every strongly sequentially compact space is sequentially compact.*

Proof. The subspace $X = \{\{x\} \in \mathcal{K}(X) : x \in X\} \subset \mathcal{K}(X)$ is a closed subset of the sequentially compact $\mathcal{K}(\hat{X})$ and X is homeomorphic to X. Since $\mathcal{K}(X)$ is sequentially compact, so is X.

The converse is not necessarily true. There exist spaces which are sequentially compact but not strongly sequentially compact. Such an example will be given after the following lemma.

LEMMA 4.3. [**1**]. *If a compact space Y is a continuous image of the remainder $cX - z(X)$ of a compactificatioin cX a locally compact space X, then the space X has a compactification $\acute{c}X \leq cX$ with the remainder homeomorphic to Y.*

Applying transfinite induction we get a continuous mapping $f : \beta N - N \to [0, \omega_1]$ onto the space $[0, \omega_1]$ of all ordinal numbers less than or equal to the first uncountable ordinal ω_1 [**1**, p. 296].

Applying Lemma 3.3, there exists a compactification gN of the space N whoose remainder $gN - N$ coincides with $[0, \omega_1]$. The space $gN - \omega_1$ is a separable, locally compact, first countable and sequentially compact normal space which is not a compact space. Since the space $X = gN - \{\omega\}$ is separable it is not strongly countably compact. Suppose that $\mathcal{K}(X)$ is sequentially compact. Then $\mathcal{K}(X)$ is countably compact and by Theorem 2.2, $X = gN - \{\omega_1\}$ would be hypercountably compact which is impossible. Therefore, X is an example of a sequentially compact space, which is not strongly sequentially compact.

Remarks. (a) The space βN is not sequentialy compact. This can be also proved directly: to sequence $1, 2, \dots,$ of point of βN does not contain any convergent subsequence, since for every increasing sequence $k_1 < k_2 < \dots$ of positive integers the sets $A = \{k_1, k_3, \dots\}$ and $B = \{k_2, k_4, \dots\}$ have disjoint closures in βN and this implies that the sequence $k_1, k_2, k_3, \dots$ does not converge.

(b) By Proposition 2.8 the space $X = \beta N \times [0, \omega_1)$ is a hypercountably compact but is not sequentially compact because $\beta N \times \{x_0\}$, $x_0 \in [0, \omega_1)$ is a closed subspace of X homeomorphic to βN.

(c) The space $X = \beta N \times X_4$ (X_4 is the space in example 2.1) is strongly countably compact which is not hypercountably compact. By (b) the space X is not sequentially compact.

THEOREM 4.4. *Every strongly sequentially compact space is hypercountably compact.*

Proof. Let $\mathcal{K}(X)$ be a sequentially compact space. Then it is countably compact. By Theorem 2.2, X is an hcc space. This completes the proof.

PROPOSITION 4.5. *A closed subset of a strongly sequentially compact space is itself strongly sequentially compact.*

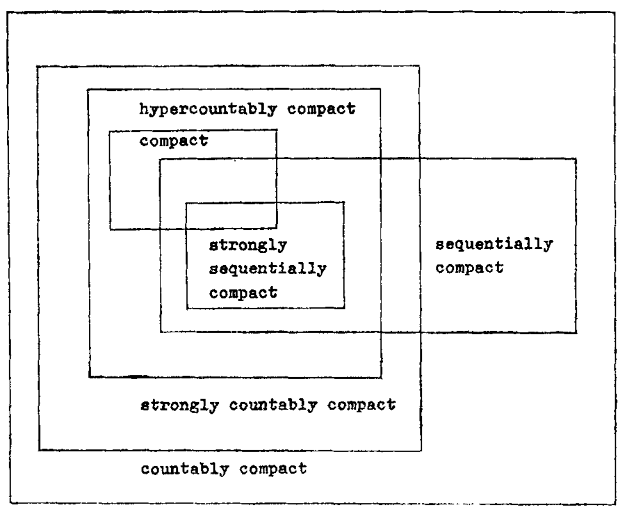
Proof. Let X be strongly sequentially compact and Y be a closed subset of X. Clearly $\mathcal{K}(Y) = \mathcal{K}(X) \cap \exp(Y)$. By [**11**], $\exp(Y)$ is a closed subset of $\exp(X)$. Hence $\mathcal{K}(Y)$ is a closed subset of $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ is sequentially compact. By definition 4.1, Y is strongly sequentially compact. This completes the proof.

Remarks. (a) The space βN is compact (hypercountably compact) but not strongly sequentially compact.

(b) The space $X = [0, \omega_1)$ is strongly sequentially compact but not compact.

(c) The space $X = [0, \omega_0]$ is both compact and strongly sequentially compact.

The following diagram ilustrates the implications which exist among the properties of compactness that we consider:



REFERENCES

[1] R. Engelking, *General Topology*, PWN, Warszava, 1977.

[2] S.P. Franklin, *Spaces in which sequences suffice*, Fund. Math. **57** (1965), 107–115.

[3] Z. Frolik, *Sums of ultrafilters*, Bull. Amer. Math. Soc. **73** (1967), 83–91.

[4] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, N. J., 1960.

[5] J. Keesling, *Normality and properties related to compactness in hyperspaces*, Proc. Amer. Math. Soc. **24** (1970), 760–766.

[6] J. Kelley, *General Topology*, Van Nostrand, Princeton, N. J., 1955.

[7] K. Kunen, *Some points in* $\beta N - N$, Math. Proc. Cambridge Phil. Soc. **80** (1976), 385–398.

[8] K. Kunen, *Weak P-points in* N^*, Proc. Bolyái János Soc. Colloq. on Topology (Budapest, 1978), vol. 2. 741–750.

[9] К. Куратовски, *Топология I и II*, Мир, Москва, 1969.

[10] M. Marjanović, *Topologizing the hypersets*, Publ. Inst. Math. (Beograd) **11(25)** (1971), 123–131.

[11] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.

[12] M.E. Rudin, *Partial orders on the types of* βN, Trais. Amer. Math. Soc. **155** (1971), 353–362.

[13] R.C. Walker, *The Stone-Čech Compactification*, Spri. ger-Verlag, New York, 1974.

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