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HIGHER-LEVEL SEQUENT-SYSTEMS FOR INTUITIONISTIC MODAL LOGIC

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Abstract. This paper presents higher-level sequent-systems for intuitionistic analogues of $S5$ and $S4$. As in [**3**] rules for modal constants involve sequents of level 2, i.e. sequents having collections of ordinary sequents of level 1 on the left and right of the turnstile. Starting from a canonical higher-level sequent formulation of $S5$, the restriction of sequents of level 2 to those with the single-conclusion property produces $S4$, without changing anything else. A similar restriction on sequents of level 1 produces Heyting $S5$, and if this restriction is made on sequents of both level 1 and 2, we obtain Heyting $S4$. The paper contains a brief discussion of Kripke-style models for the intuitionistic propositional modal logics in question.

In [**3**] we have presented sequent formulations of the modal logics $S5$ and $S4$ based on sequents of higher levels: sequents of level 1 are like ordinary sequents, sequents of level 2 have collections of sequents of level 1 on the left and right of the turnstile, etc. The rules we gave for modal constants involved sequents of level 2, whereas rules for other customary logical constants of first-order logic involved only sequents of level 1. In this paper we shall show how starting from the same sequent rules of higher level we can obtain sequent formulations of intuitionistic analogues of $S5$ and $S4$ based on the necessity operator $\Box$.

We say that a sequent has the single-conclusion property if its right-hand side is either a singleton or empty. It is well-known from [**7**] that it is possible to obtain a sequent formulation of Heyting's logic out of a sequent formulation of classical logic just by restricting sequents of level 1 to those with the single-conclusion property, without changing anything else. In [**3**] we have shown that the same restriction applied to sequents of level 2 can produce a sequent formulation of $S4$ out of a sequent formulation of $S5$, without changing anything else. Here we shall show that starting from the same sequent rules with sequents of level 1 restricted to single-conclusion sequents, if we make no restriction upon sequents of level 2, we obtain a sequent formulation of an intuitionistic analogue of $S5$, which we shall call $S5_H$. If we also restrict sequents of level 2 to singe-conclusion sequents,

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without changing anything else, we obtain a sequent formulation of an intuitionistic analogue of $S4$, which we shall call $S4_H$. This is shown succintly in the following table:

	$S5$	$S4$	$S5_H$	$S4_H$
level 1			restricted	restricted
level 2		restricted		restricted

To simplify matters we shall concentrate in this paper on propositional logic. Only in the final, fifth section we shall indicate briefly how to obtain sequent-systems for first-order predicate logics corresponding to $S5_H$ and $S4_H$. In the first section we shall present our sequent-systems, and in the second section we shall show to what Hilbert-style systems these sequent-systems correspond. In the third section we shall briefly consider Kripke-style models for our intuitionistic propositional modal logics. In the fourth section we shall consider alternative bases, with strict implication $\prec$ and the possibility operator $\Diamond$, for our intuionistic modal logics. This paper is a companion to [**3**], [**1**] and [**4**], and we assume the reader is acquainted with these papers. However, to make this paper more self-contained we shall briefly recapitulate some basic notions.

1. Sequent-systems. Let O be the language of modal propositional logic based on the connectives $\rightarrow, \wedge, vee, \bot, \top$ and $\Box$. We use $A, B, \dots, A_1, \dots$ as schematic letters for formulae of O. We define $\neg A$ as $A \rightarrow \bot$, and $A \leftrightarrow B$ as $(A \rightarrow B) \wedge (B \rightarrow A)$. Parentheses are omitted with the assumption that $\rightarrow$ and $\leftrightarrow$ bind more strongly than $\wedge$ and $\vee$.

Starting from O we build the language D as follows. The formulae of O are formulae of level 0. The empty set sign $\emptyset$ is a set term of any level ≥ 1. Let $A_1^n, \dots, A_k^n$, where $k \geq 1$, be distinct formulae of level n; then $\{A_1^n, \dots, A_k^n\}$ is a set term of level $n+1$. If Γ and Δ are set terms of level n, then $\Gamma \vdash^n \Delta$ is a formula of level n, called a sequent. The set term $\Gamma \cup \Delta$ stands for the union of the sets Γ and Δ, and the set term $\Gamma + \Delta$ for the disjoint union of Γ and Δ (for details see [**3**,§1]).

A sequent $\Gamma \vdash^n \Delta$ has the *single-conclusion* property iff Δ is either a singleton or $\emptyset$. A rule is of level n iff the highest level of formulae occurring in it is n. A rule is *level preserving* iff all formulae occurring in it are of the same level. The *horizontalization* of a level-preserving rule

$$\frac{\Psi}{A^n}$$

is the sequent $\Gamma \vdash^{n+1} \{A^n\}$ where Γ is obtained from the set of premises by omitting repetitions of formulae.

For our sequent-systems we shall asume the following *structural* rules:

Ascending **(A)** $\dfrac{A^n}{\emptyset \vdash^{n+1} \{A^n\}}$

Descending **(D)** $\dfrac{\emptyset \vdash^{n+1} \{A^n\}}{A^n}$

Iteration **(I)** $\dfrac{A^n}{A^n}$

Cut **(C)** $\dfrac{\Gamma_1 \vdash^{n+1} \Delta_1 + \{A^n\} \quad \Gamma_2 + \{A^n\} \vdash^{n+1} \Delta_2}{\Gamma_1 \cup \Gamma_2 \vdash^{n+1} \Delta_1 \cup \Delta_2}$

Thinnrng **(T)** $\dfrac{\Gamma \vdash^{n+1} \Delta}{\Gamma \cup \Gamma_1 \vdash^{n+1} \Delta \cup \Delta_1}$

In all these rules we have $n \geq 0$. An instance of a rule R of level n will be denoted by R^n.

Let the *double-line* rule

$$\frac{B_1^n \ldots B_k^n}{\overline{A^n}}$$

be an abbreviation for the following list of rules:

$$\frac{B_1^n \ldots B_k^n}{A^n}, \ \frac{A^n}{B_1^n}, \ldots, \frac{A^n}{b_k^n}.$$

If R is the name of this double-line rule, $R\downarrow$ is the name of the first rule in the list, and $R\uparrow$ designates any of the remaining rules in the list. For the connectives of O we shall give the following double-line rules:

$(\rightarrow)$ $$\frac{\Gamma + \{A\} \vdash^1 \Delta + \{B\}}{\overline{\Gamma \vdash^1 \Delta + \{A \rightarrow B\}}}$$

$(\wedge)$ $$\frac{\Gamma \vdash^1 \Delta + \{A\} \qquad \Gamma \vdash^1 \Delta + \{B\}}{\overline{\Gamma \vdash^1 \Delta + \{A \wedge B\}}}$$

$(\vee)$ $$\frac{\Gamma + \{A\} \vdash^1 \Delta \qquad \Gamma + \{B\} \vdash^1 \Delta}{\overline{\Gamma + \{A \vee B\} \vdash^1 \Delta}}$$

$(\bot)$ $$\frac{\Gamma \vdash^1 \emptyset}{\overline{\Gamma \vdash^1 \{\bot\}}}$$

$(\top)$ $$\frac{\emptyset \vdash^1 \Delta}{\overline{\{\top\} \vdash^1 \Delta}}$$

$(\Box)$ $$\frac{\Pi + \{\emptyset \vdash^1 \{A\}\} \vdash^2 \Sigma + \{\Gamma \vdash^1 \Delta\}}{\overline{\Pi \vdash^2 \Sigma + \{\Gamma + \{\Box A\} \vdash^1 \Delta\}}}$$

Axioms for our sequent-systems will be systematically generated from the rules by assuming the horizontalizations of all the level-preserving rules, i.e. all the rules mentioned above except **A** and **D**.

The sequent-system $DS5$ is obtained from the rules above, and the corresponding horizontalizations, in the full language D. If we assume these rules and horizontalizations in a restricted version of D where sequents of level 2 must have the single-conclussion property, everything else remaining unchanged, we obtain the sequent-system DS_4. The sequent-system $DS5_H$ is obtained by an analogous restriction for sequents of level 1 instead of sequents of level 2, and for $DS4_H$ we make this restriction for sequents of both level 1 and level 2. According to Lemma 4 of [**3**] sequents of all levels higher than 2 are irrelevant for $DS5_H$ and $DS4_H$, as well as for $DS5$ and $DS4$.

2. Hilbert-style systems. Let H be the Heyting propositional calculus axiomatized by the following rule and axiom-schemata:

1. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$, 3. $A \to (B \to A)$,

4. $(C \to A) \to ((C \to B) \to (C \to A \wedge B))$, 5. $A \wedge B \to A$,

6. $A \wedge B \to B$,

7. $A \to A \vee B$, 8. $B \to A \vee B$, 9. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$

10. $\bot \to A$, 11. $\top$.

The classical propositional calculus C is obtained from H by adding the axiom-schema:

$$12. A \vee (A \to B).$$

The system $S4_H$ is obtained from H by adding the following rule and axiom-schemata:

$\Box$ 1. $\dfrac{A}{\Box A}$,

$\Box$ 2. $\Box(A \to B) \to (\Box A \to \Box B)$, $\Box$ 3. $\Box A \to A$, $\Box$ 4. $\Box A \to \Box\Box A$.

The system $S5_H$ is obtained from $S4_H$ by adding the axiom-schema:

$$\Box 5.\ \Box A \vee \Box(\Box A \to B).$$

The systems $S4$ and $S5$ are extensions of C obtained from $S4_H$ and $S5_H$ respectively by adding the axiom-schema 12.

The system $S4_H$ is what everybody would take as the natural intuitionistic analogue of $S4$. This system has been considered in a number of works (see [**4**] and [**1**] for references). On the other hand, there are various nonequivalent systems which may be considered as intuitionistic analogues of $S5$. The system $S5_H$, axiomatized equivalently with $\Box A \vee \Box\neg\Box A$ rather than our $\Box 5$, has been considered in [**2**], [**9**], [**10**], [**5**], [**6**] and [**4**] (in these papers one can find also a number of intuitionistic analogues of $S5$).

From results concerning Kripke-style models of Section 3, it follows easily that the containment relations between our modal systems, as shown in the following chart:

$$\begin{tikzcd}
 & \mathbf{S5} & \\
\mathbf{S5_H} \arrow[ur] & & \mathbf{S4} \arrow[ul] \\
 & \mathbf{S4_H} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

are proper. With these models it is also easy to show that $S4_H$ and $S5_H$ are conservative extensions of the nonmodal system H. The systems $S4$ and $S5$ are, of course, decidable, and the same holds for $S4_H$ and $S5_H$ (see [**9**] and [**10**]). It is easy to show that $S4_H$ has the disjunction property, whereas it is clear that $S5_H$ lacks this property (see [4]). So, $S5_H$ may perhaps be considered "intuitionistically spurious".

It follows from the results of [**3**] that a formula A of O is provable in $DS4$ (respectively $DS5$) iff A is provable in $S4$ (respectively $S5$). Here we shall demonstrate the following analogous theorem:

THEOREM 1.1. *A formula A of O is provable in $DS4_H$ iff A is provable in $S4_H$.*

1.2. *A formula A of O is provable in $DS5_H$ iff A is provable in $S5_H$.*

This theorem will follow from the following lemmata:

LEMMA 1.1. *The rules of $S4_H$ are derivable and the axioms of $S4_H$ are provable in $DS4_H$.*

1.2. *The axiom-schema $\Box 5$ is provable in $DS5_H$.*

Proof: The proof of 1.1 is exactly as in [**3**] (see Lemma 7.1). For 1.2 we have

$$\text{(a)} \quad \{\emptyset \vdash^1 \{A\}\} \vdash^2 \{\emptyset \vdash^1 \{\Box A\}\}$$

which is proved as in [**3**, p. 159]. Then we proceed as follows:

$$\cfrac{T^2\ \cfrac{\{\emptyset \vdash^1 \{A\}\} \vdash^2 \{\emptyset \vdash^1 \{\Box A\}\}}{\{\emptyset \vdash^1 \{A\}\} \vdash^2 \{\emptyset \vdash^1 \{\Box A\}, \emptyset \vdash^1 \{B\}\}}}{\emptyset \vdash^2 \{\emptyset \vdash^1 \{\Box A\}, \{\Box A\} \vdash^1 \{B\}\}}\ (\Box)\downarrow$$

which by applying the horizontalization of $(\to)\downarrow$, an instance of (a), and C^2 yields $\emptyset \vdash^2 \{\emptyset \vdash^1 \{\Box A\}, \emptyset \vdash^1 \{\Box(\Box A \to B)\}\}$. From that with $(\vee)\uparrow$, C^2, D^2 and D^1 we obtain $\Box A \vee \Box(\Box A \to B)$. q.e.d.

For the following translation from D into O:

$o(A)$ is A,

$\hat{o}(\Gamma)$ is $\begin{cases} o(A_1^n) \wedge \cdots \wedge o(A_k^n) \ \text{if} \ \Gamma = \{A_1^n, \ldots, A_k^n\}, \ k \geq 1 \\ \top \ \text{if} \ \Gamma = \emptyset, \end{cases}$

$\check{o}(\Gamma)$ is $\begin{cases} o(A_1^n) \vee \cdots \vee o(A_k^n) \ \text{if} \ \Gamma = \{A_1^n, \ldots, A_k^n\}, \ k \geq 1 \\ \bot \ \text{if} \ \Gamma = \emptyset, \end{cases}$

$o(\Gamma \vdash^{n+1} \Delta)$ is $\Box(\hat{o}(\Gamma) \to \hat{o}(\Delta))$

we have:

LEMMA 2.1. *If A^n is provable in $DS4_H$, then $o(A^n)$ is provable in $S4_H$.*

2.2. *If A^n is provable in $DS5_H$, then $o(A^n)$ is provable in $S5_H$.*

Proof. 2.1. By induction on the length of proof of A^n in $DS4_H$. For **A** and **D** we use:

$$\frac{A}{\overline{\Box(\top \to A)}}.$$

For **I** we use $\Box(A \to A)$. For **C** we use $\Box(G_1 \to D_1 \vee A) \wedge \Box(G_2 \vee A \to D_2) \to \Box(G_1 \wedge G_2 \to D_1 \vee D_2)$. For **T** we use $\Box(G \to D) \to \Box(G \wedge G_1 \to D \vee D_1)$. For $(\to)$ we use $\Box(G \wedge A \to B) \leftrightarrow \Box(G \to (A \to B))$. For $(\wedge)$ we use $\Box(G \to A) \wedge \Box(G \to B) \leftrightarrow \Box(G \to A \wedge B)$. For $(\vee)$ we use $\Box(G \wedge A \to D) \wedge \Box(G \wedge B \to D) \leftrightarrow \Box(G \wedge (A \vee B) \to D)$. For $(\bot)$ and $(\top)$ we use $A \to A$. Finally, for $(\Box)$ we use:

$$\Box(\Box P \wedge \Box A \to \Box(G \to D)) \leftrightarrow \Box(\Box P \to \Box(G \wedge \Box A \to D)).$$

To prove this last formula in $S4_H$ from left to right we proceed as follows:

$$\begin{array}{l}
\Box(\Box P \wedge \Box A \to \Box(G \to D)) \\ \hline
\Box(\Box P \wedge \Box A \to (G \to D)) \\ \hline
\Box(\Box P \to (G \wedge \Box A \to D)) \\ \hline
\Box\Box(\Box P \to (G \wedge \Box A \to D)) \\ \hline
\Box(\Box P \to \Box(G \wedge \Box A \to D))
\end{array}$$

The converse is obtained analogously.

2.2. We proceed as for 2.1. The only additional case is for $(\Box)$, where we use:

$$\Box(\Box P \wedge \Box A \to \Box C \vee (G \to D)) \leftrightarrow \Box(\Box P \to \Box C \vee \Box(G \wedge \Box A \to D)).$$

To prove this formula in $S5_H$ from left to right we show first that $\Box(\Box A \vee B) \to \Box A \vee \Box B$ is provable in $S5_H$. We have:

$$\Box(\Box A \to B) \to (\Box(\Box A \vee B) \to \Box A \vee \Box B) \text{ and}$$
$$\Box A \to (\Box(\Box A \vee B) \to \Box A \vee \Box B)$$

and we apply $\Box 5$. Then we proceed as follows:

$$\dfrac{\dfrac{\Box(\Box P \wedge \Box A \to \Box C \vee \Box(G \to D))}{\Box(\Box A \to (\Box P \to \Box C \vee \Box(G \to D)))} \qquad \dfrac{\Box(\Box A \vee \Box(\Box A \to \Box(G \to D)))}{\Box(\Box A \vee (\Box A \to \Box(G \to D)))}}{\begin{array}{l}
\Box((\Box P \to \Box C \vee \Box(G \to D)) \vee (\Box A \to \Box(G \to D))) \\ \hline
\Box(\Box P \to \Box C \vee (\Box A \to \Box(G \to D))) \\ \hline
\Box(\Box P \to \Box C \vee (G \wedge \Box A \to D)) \\ \hline
\Box\Box(\Box P \to \Box C \vee (G \wedge \Box A \to D)) \\ \hline
\Box(\Box P \to \Box C \vee \Box(G \wedge \Box A \to D))
\end{array}}\ .$$

For the last step we use $\Box(\Box C \vee (G \wedge \Box A \to D)) \to \Box C \vee \Box(G \wedge \Box A \to D)$. This proves our formula from left to right. The converse is quite straightforward. q.e.d.

From Lemmata 1.1, 1.2, 2.1 and 2.2 we easily infer Theorem 1.1 and 1.2.

3. Kripke-style models. For $S4_H$ and $S5_H$ we shall now give Kripke-style models based on two accessibility relations, one intuitionistic and the other modal. These models are dealt with extensively in [**1**] and [**4**]. We shall first recapitulate briefly the basic notions we need.

An $H\Box$ frame is $\langle X, R_I, R_M \rangle$ where $X \neq \emptyset$, $R_I \subseteq X^2$ is reflexive and transitive, $R_M \subseteq X^2$, and $R_I R_M \subseteq R_M R_I$ ($R_1 R_2$ is short for $R_1 \circ R_2$). The variables $u, v, w, u_1 \ldots$ range over X. An $H\Box$ model is $\langle X, R_I, R_M, V \rangle$ where $\langle X, R_I, R_M \rangle$ is an $H\Box$ frame and the valuation V is a mapping from the set of propositional variables of O to the power set of X such that the following Heredity Condition is satisfied for every propositional variable p:

$$\forall u_1, u_2 (u_1 R_I u_2 \Rightarrow (u_1 \in V(p) \Rightarrow u_2 \in V(p))).$$

The relation $\models$ in $u \models A$ is defined as usual, except that for $\rightarrow$ it involves R_I as in intuitionistic Kripke models, whereas for $\Box$ it involves R_M as in modal Kripke models. A formula A holds in an $H\Box$ model iff $(\forall u \in X) u \models A$; and A holds in a frame Fr (i.e. $Fr \models A$) iff A holds in every model with this frame. An $H\Box$ frame is condensed iff $R_I R_M = R_M$, and it is strictly condensed iff $R_I R_M = R_M R_I = R_M$. We use $R_\Box$ as an abbreviation for $R_M R_I$.

In [**1**] and [**4**] one can find a proof of the following statements, for every $H\Box$ frame Fr:

all theorems of H plus $\Box 1$ and $\Box 2$ hold in Fr,

$Fr \models \Box A \rightarrow A$ iff $R_\Box$ is reflexive,

$Fr \models \Box A \rightarrow \Box\Box A$ iff $R_\Box$ is transitive.

Here we shall prove the following lemma:

LEMMA 3. *For every $H\Box$ frame Fr we have Fr $\models \Box A \vee \Box(\Box A \rightarrow B)$ iff $R_\Box^{-1} R_\Box \subseteq R_\Box$ (i.e. $R_\Box$ is Euclidean).*

Proof. ($\Rightarrow$) Suppose *not* $R_\Box^{-1} R_\Box \subseteq R_\Box$, i.e. for some v, v_1 and v_2 we have $v R_\Box v_1$ it and $v R_\Box v_2$ *and not* $v_1 R_\Box v_2$. Next let $\forall u (u \models p \Leftrightarrow v_1 R_\Box u)$ and $\forall u$ *not* $u \models q$. It is easy to show that there is an $H\Box$ model such that this is satisfied. In this model *not* $v_2 \models p$, and hence *not* $v \models \Box p$. Also, $v_1 \models \Box p$ and *not* $v_1 \models q$. Hence, *not* $v \models \Box(\Box p \rightarrow q)$, and so we obtain *not* $v \models \Box p \vee \Box(\Box p \rightarrow q)$.

($\Leftarrow$) Suppose *not Fr* $\models \Box A \vee \Box(\Box A \rightarrow B)$. So, for some v we have *not* $v \models \Box A$ *and not* $v \models \Box(\Box A \rightarrow B)$. Hence, there is a v_2 such that $v R_\Box v_2$ *and not* $v_2 \models A$, and there is a v_1 such that $v R_\Box v_1$ and $v_1 \models \Box A$ *and not* $v_1 \models B$. But from the euclideanity of $R_\Box$ it follows that $v_1 R_\Box v_2$, and since $v_1 \models \Box A$ *and not* $v_2 \models A$ we obtain a contradiction. q.e.d.

In [**4**] one can find a proof of the following statement: A is provable in $S4_H$ iff A holds in every (condensed, strictly condensed) $H\Box$ frame where $R_\Box$ is reflexive and transitive. Analogously, we can establish that A is provable in $S5_H$ iff A holds in every condensed, strictly condensed) $H\Box$ frame where $R_\Box$ is reflexive, transitive and Euclidean.

Of course, $R_\Box$ is reflexive, transitive and Euclidean iff $R_\Box$ is reflexive, transitive and symmetric. Since $Fr \models A \vee \Box(\Box A \to B)$ iff $R_\Box$ is symmetric, the axiom-schema $A \vee \Box(\Box A \to B)$ can replace $\Box 5$ in $S5_H$. As $\Box 5$, the axiom-schema $\Box A \vee \Box\neg\Box A$ corresponds to the euclideanity of $R_\Box$, and similarly, $A \vee \Box\neg\Box A$ corresponds to the symmetry of $R_\Box$.

4. Alternative bases. As stated in [**3**], if the strict implication $A \prec B$ is defined as $\Box(A \to B)$, in $DS4_H$ and in all stronger sequent-systems we can derive the rules:

$$(\prec) \qquad \frac{\Pi + \{\{A\} \vdash^1 \{B\}\} \vdash^2 \Sigma + \{\Gamma \vdash^1 \Delta\}}{\Pi \vdash^1 \Sigma + \{\Gamma + \{A \prec B\} \vdash^1 \Delta\}}$$

and their horizontalizations. Alternatively, we could base O and our modal systems in D on the connective $\prec$ and the rules $(\prec)$. Then by defining $\Box A$ as $\top \prec A$ we can derive the rules $(\Box)$ and their horizontalizations.

If the possibility operator is defined by $\Diamond A =_{\mathrm{df}} \neg\Box\neg A$, then in $DS5_H$ and $DS4_H$ we can derive the rules:

$$(\Diamond) \qquad \frac{\Pi + \{\{A\} \vdash^1 \emptyset\} \vdash^2 \Sigma + \{\Gamma \vdash^1 \Delta\}}{\Pi \vdash^2 \Sigma + \{\Gamma \vdash^1 \Delta + \{\Diamond A\}\}}$$

and their horizontalizations. These rules are now available only with Δ empty, whereas with $DS5$ and $DS4$ it need not be empty. However, $\Diamond$ and the double--line rule $(\Diamond)$ cannot serve as an alternative basis for $DS5_H$ and $DS4_H$. Take the systems exactly like $DS5_H$ and $DS4_H$ save that $\Box$ is replaced by $\Diamond$, $(\Box)$ is replaced by $(\Diamond)$, and $\Box A$ is defined as $\neg\Diamond\neg A$. In these systems we cannot prove $\Box A \to A$. Otherwise, in the system $S5_H$ we could prove $\neg\neg\Box\neg\neg A \to A$, and it is easy to construct an $H\Box$ model with $R_\Box$ reflexive, transitive and Euclidean such that $\neg\neg\Box\neg\neg A \to A$ is falsified in this model (cf. [**6**], [**8**]).

5. First-order modal logic. It is not difficult to find along the lines of [**3**], higher-level sequent formulations of first-order predicate logics corresponding to $S5_H$ and $S4_H$. If O is now the language of the first-order predicate calculus with the constants $\to, \wedge, \vee, \bot, \top, \forall, \exists$ (and eventually $=$), we can proceed exactly as in [**3**] to obtain first-order sequent-systems corresponding to $DS5_H$ and $DS4_H$. The additional rules are Substitution for individual variables and two double-line rules for quantifiers:

$$\left.\begin{array}{ll}(\forall) & \dfrac{\Gamma \vdash^1 \Delta + \{A(x)\}}{\Gamma \vdash^1 \Delta + \{\forall x A(x)\}} \\[2ex] (\exists) & \dfrac{\Gamma + \{A(x)\} \vdash^1 \Delta}{\Gamma + \{\exists x A(x)\} \vdash^1 \Delta}\end{array}\right\} \text{provided } x \text{ is not free in } \Gamma \text{ or } \Delta$$

(and eventually an additional double-line rule for identity), where none of these rules is subject to horizontalization. The restrictions made upon D to obtain first--order sequent-systems corresponding to $DS5_H$ and $DS4_H H$ are as in Section 1. It is a straightforward matter to obtain Hilbert-style systems corresponding to these first-order sequent-systems.

It is easy to show that the Barcan Formula $\forall x\Box A \to \Box\forall xA$ is provable in the first-order sequent-system corresponding to $DS5_H$: since the formulae $\neg\Box\neg\Box A \to A$ and $A \to \Box\neg\Box\neg A$ are provable in $DS5_H$ (the first of these schemata can replace $\Box 5$ for the axiomatization of $S5_H$), we have

$$\begin{array}{l}
\forall x\Box A \to \Box A \\ \hline
\neg\Box\neg\forall x\Box A \to \neg\Box\neg\Box A \\ \hline
\neg\Box\neg\forall x\Box A \to A \\ \hline
\neg\Box\neg\forall x\Box A \to \forall xA \\ \hline
\Box\neg\Box\neg\forall x\Box A \to \Box\forall xA \\ \hline
\forall x\Box A \to \Box\forall xA\,.
\end{array}$$

A fortiori, the Barcan Formula is provable in the first-order sequent-system corresponding to $DS5$. This formula is not provable in the first-order sequent--system corresponding to $DS4$, and hence it cannot be provable in the first-order sequent-system corresponding to $DS4_H$.

Considerations concerning the uniqueness of characterization of $\Box$ from [**3**] apply in the present context too.

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