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CONDITIONAL PROBABILITY IN NONSTANDARD ANALYSIS

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Abstract. In this paper we apply the theory of Loeb measure to conditional probability for hyperfinite Loeb spaces. We show that conditional probability $^{\sim}P(\cdot/A)$ on a Loeb space $(V, \mathfrak{M}(^{\sim}P), ^{\sim}P)$ for $A \in^{*} \mathfrak{P}(V)$ $(P(A) > 0$ and $P(A) \approx' 0$[1] is a Loeb measure and for $A \in \mathfrak{M}(^{\sim}P)$ $(^{\sim}P(A) > 0)$ can be represented by a Loeb measure. For the case $A \in \mathfrak{M}(^{\sim}P)$ we prove that there exists a set $C \in^{*} \mathfrak{P}(V)$ such that $^{\sim}P(\cdot/A)$ is equal to the Loeb conditional probability $L(P(\cdot/C))$. We introduce internal conditional probability relative to an internal subalgebra $\mathfrak{A}$ of $^{*}\mathfrak{P}(V)$ as in case of finite standard probability spaces. We show, analogously to a well-known probability result, that internal conditional probability $P(A/\mathfrak{A})$, $A \in^{*} \mathfrak{P}(V)$, and internal conditional expectation $E(X/\mathfrak{A})$, X is S-integrable, are P-a. s. unique, in nonstandard sense, random variables on $(V, \mathfrak{A}, P)$. Finally, we give a nonstandard characterization of conditional probability $^{\sim}P(A/\mathfrak{M}(\mathfrak{A}))$, $A \in \mathfrak{M}(^{\sim}P)$ on a Loeb space $(V, \mathfrak{M}(^{\sim}P), ^{\sim}P)$. We prove that there exists a set $C \in^{*} \mathfrak{P}(V)$ such that $P(C/\mathfrak{A})$ is the lifting of $^{\sim}P(A/\mathfrak{M}(\mathfrak{A}))$.

Introduction. In this paper we concern ourselves with conditional probability for hyperfinite Loeb spaces. We use the well-known results from the theory of Loeb measure [**8**] and nonstandard probability [**2**], [**10**] and the methodology developed by P. Loeb, J. Keisler, R. Anderson and others.

In the first part we define internal conditional probability $P(\cdot/A)$, $A \in {}^{*}\mathfrak{P}(V)$ for a hyperfinite probability space $(V, {}^{*}\mathfrak{P}(V), P)$ and give the nonstandard representation of conditional probability $^{\sim}P(\cdot/A)$, $A \in \mathfrak{M}(^{\sim}P)$ on the Loeb space $(V, \mathfrak{M}(^{\sim}P), ^{\sim}P)$. We show that for $A \in^{*} \mathfrak{P}(V)$ with $P(A) > 0$ and $P(A) \approx' 0$ $^{\sim}P(\cdot/A)$ is a Loeb measure on $(V, \mathfrak{M}(^{\sim}P))$ and for $A \in \mathfrak{M}(^{\sim}P)$ with $^{\sim}P(A) > 0$ there exists a set $C \in^{*} \mathfrak{P}(V)$ such that $^{\sim}P(\cdot/A)$ can be represented by the Loeb conditional probability $L(P(\cdot/C))$.

In the second part we define, analogously to the definition of internal conditional expectation $E(X/\mathfrak{A})$, [**10**], internal conditional probability $P(A/\mathfrak{A})$ $A \in {}^{*}\mathfrak{P}(V)$, is an internal subalgebra of ${}^{*}\mathfrak{P}(V))$ for a hyperfinite probability space $(V, {}^{*}\mathfrak{P}(V), P)$. We show that so-introduced $P(A/\mathfrak{A})$ $(E(X/\mathfrak{A})$ as well) is

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[1] $\approx'$ is the negation of $\approx$

P-a. s. unique random variable on $(V, \mathfrak{A}, P)$. The P-a. s. uniqness of the $P(A/\mathfrak{A})$ $(E(X/\mathfrak{A}))$ is introduced in theorem 4, corresponding to the same concept from standard probability. Finally, we give a nonstandard characterization of conditional probability $P(A/\mathfrak{M}(\mathfrak{A}))$ $(A \in \mathfrak{M}(^{\sim}P)$, $(A \in \mathfrak{M}(^{\sim}P)$ $\mathfrak{M}(\mathfrak{A}$ is a sub-σ-algebra of $\mathfrak{M}(^{\sim}P))$ on a Loeb space $(V, \mathfrak{M}(^{\sim}P), {}^{\sim}P)$. We show that for $A \in \mathfrak{M}(^{\sim}P)$ there exists a set $B \in^* \mathfrak{P}(V)$ such that $P(B/\mathfrak{A})$ is lifting of ${}^{\sim}P(A/\mathfrak{M}(\mathfrak{A}))$.

We assume $(V, {}^*\mathfrak{P}(V), P)$ to be a hyperfinite probability space and $(V, \mathfrak{M}(^{\sim}P), {}^{\sim}P)$ a Loeb space constructed from it, i.e. Loeb measure ${}^{\sim}P$ is defined by

$$\begin{aligned} {}^{\sim}P(F) &= \inf\{\mathrm{st}(P(A)) \mid F \subseteq A \text{ and } A \in^* \mathfrak{P}(V)\} \\ &= \sup\{\mathrm{st}(P(A)) \mid A \subseteq F \text{ and } A \in^* \mathfrak{P}(V)\} \end{aligned} \qquad \text{for } F \subseteq V$$

and $\mathfrak{M}(^{\sim}P)$ is a σ-algebra of all ${}^{\sim}P$-measurable sets $F \subseteq V$.

According to standard probability, for $A \in^* \mathfrak{P}(V)$ with $P(A) > 0$ we define internal conditional probability $P(\cdot/A)$ of an internal event B relative to A by

$$P(B/A) = P(A \cap B)/P(A)$$

It is easy to show that $(V, {}^* B(V), P(\cdot/A))$ is a hyperfinite probability space, so it gives rise to a Loeb space denoted by $(V, \mathfrak{M}(L(P(\cdot/A))), L(P(\cdot/A)))$.

On the other hand, for a Loeb space $(V, \mathfrak{M}(^{\sim}P), {}^{\sim}P)$, conditional probability ${}^{\sim}P(\cdot/A)$ of an event $B \in \mathfrak{M}(^{\sim}P)$ relative to $A \in \mathfrak{M}(^{\sim}P)$, ${}^{\sim}P(A) > 0$, is standardly defined by

$${}^{\sim}P(B/A) = {}^{\sim}P(A \cap B)/{}^{\sim}P(A)$$

It is well known that ${}^{\sim}P(\cdot/A)$ is a probability measure on $(V, \mathfrak{M}(^{\sim}P))$ but not necessary a Loeb measure. However, for $A \in^* \mathfrak{P}(V)$ with $P(A) > 0$ $P(A) \approx' 0$, ${}^{\sim}P(\cdot/A)$ is a Loeb measure and we shall show it in this paper.

Let $\sigma(^{\sim}P(\cdot/A))$ be σ-algebra of all ${}^{\sim}P(\cdot/A)$-measurable sets, i.e.

$$\sigma(^{\sim}P(\cdot/A)) = \{F \subseteq V \mid F \cap A \in \mathfrak{M}(^{\sim}P)\}$$

It is obvious that $\mathfrak{M}(^{\sim}P) \subset \sigma(^{\sim}P(\cdot/A))$. From the theory of Loeb measure we know that ${}^{\sim}P$ is a complete measure on $(V, \mathfrak{M}(^{\sim}P))$ and that $\mathfrak{M}(^{\sim}P)$ is a completion of $Ļ(V)$ relative to ${}^{\sim}P$. We have the same for ${}^{\sim}P(\cdot/A)$:

Lemma 1. *Probability measure* ${}^{\sim}P(\cdot/A)$ *is a complete measure on* $(V, \mathfrak{M}(^{\sim}P))$ *and* $\mathfrak{M}(^{\sim}P)$ *is a completion of* $Ļ(V)$ *relative to* ${}^{\sim}P(\cdot/A)$ *in the sense that for* $F \in \mathfrak{M}(^{\sim}P)$ *there exist sets* $Z \in Ļ(V)$ *and* $N \subseteq V$ *such that*

$$F = Z \cup N, \qquad N \subseteq U, \qquad \textit{and} \qquad {}^{\sim}P(U/A) = 0.$$

Proof Let $F \in \mathfrak{M}(^{\sim}P)$, ${}^{\sim}P(F/A) = 0$ and $M \subseteq F$. Than is $F \cap A \in \mathfrak{M}(^{\sim}P)$, $M \cap A \subseteq F \cap A$ and ${}^{\sim}P(F \cap A) = 0$. Since ${}^{\sim}P$ is a complete measure, $M \cap A \in \mathfrak{M}(^{\sim}P)$ and ${}^{\sim}P(M \cap A) =)$. This implies that M is ${}^{\sim}P(\cdot/A)$ – measurable and ${}^{\sim}P(M/A) = 0$.

Let $F \in \mathfrak{M}(^\sim P)$. Then there exist sets $Z \in \text{Ļ}(V)$ and $N \subseteq V$ such that $F = Z \cup N$, $N \subseteq U$ and $^\sim P(U) = 0$. Since $U \cap A \subseteq U$ and $^\sim P$ is a complete measure, it follows that $^\sim P(U \cap A) = 0$, i.e. $^\sim P(U/A) = 0$. Hence $\mathfrak{M}(^\sim P)$ is a completion of $\text{Ļ}(V)$ relative to $^\sim P(\cdot/A)$.

The next theorem shows that conditional probability $^\sim P(\cdot/A)$ for $A \in^* \mathfrak{P}(V)$, $P(A) > 0$, and $P(A) \approx' 0$ is a Loeb measure on $(V, \mathfrak{M}(^\sim P))$.

THEOREM 1. *Let* $A \in^* \mathfrak{P}(V)$, $P(A) >)$ *and* $P(A) \approx' 0$. *Then* $(V, \mathfrak{M}(^\sim P), {}^\sim P(\cdot/A))$ *is a Loeb probability space.*

Proof. We show that $^\sim P(\cdot/A)$ is a Loeb measure obtained from internal conditional probability $P(\cdot/A)$. Using notations already defined we prove that

$$L(P(B/A)) = {}^\sim P(B/A) \qquad \text{for } B \in \mathfrak{M}(^\sim P) \tag{1}$$

Let $F \in^* \mathfrak{P}(V)$. Since $P(A) > 0$ and $P(A) \approx' 0$

$$\begin{aligned} {}^\sim P(F/A) &= {}^\sim P(F \cap A)/{}^\sim P(A) = \operatorname{st}(P(F \cap A))/\operatorname{st}(P(A)) = \\ &= \operatorname{st}(P(F \cap A)/P(A)) = L(P(F/A)). \end{aligned}$$

Let $F \in \text{Ļ}(V)$. The definition of Loeb measure

$$\begin{aligned} &L(P(F/A)) = \\ &= \sup\{\operatorname{st}(P(C/A)) \mid C \in^* \mathfrak{P}(V), C \subseteq F\} = \inf\{\operatorname{st}(P(D/A)) \mid (D \in^* \mathfrak{P}(V), D \supseteq F\} \end{aligned}$$

implies that for $\varepsilon \in^\sigma R^+$ there exist sets $C, D \in^* \mathfrak{P}(V)$ such that

$$^\sim P(C/A) \leq {}^\sim P(F/A) {}^\sim P(D/A) \qquad \text{and} \tag{2}$$

$$^\sim P(D/A) - \varepsilon < L(P(F/A)) < {}^\sim P(C/A) + \varepsilon \tag{3}$$

Relations (2) and (3) imply

$$^\sim P(F/A) - \varepsilon < L(P(F/A)) < {}^\sim P(F/A) + \varepsilon \qquad \text{i.e.} \qquad L(P(F/A)) = {}^\sim P(F/A)$$

Let $F \in \mathfrak{M}(^\sim P)$. Then, according to [**8**], there exist sets $C, D \in \text{Ļ}(V)$ such that $C \subseteq F \subseteq D$ and $^\sim P(C) = {}^\sim P(F) =^\sim P(D)$. We show that $F \in \mathfrak{M}(L(P(\cdot/A)))$ and that $L(P(F/A)) = {}^\sim P(F/A)$. Since

$$\begin{aligned} {}^\sim P(DUA) &= {}^\sim P(D) + {}^\sim P(A) - {}^\sim P(D \cap A) \leq {}^\sim P(D) + {}^\sim P(A) - {}^\sim P(C \cap A) = \\ &= {}^\sim P(C) + {}^\sim P(A) - {}^\sim P(C \cap A) = {}^\sim P(C \cup A) \leq {}^\sim P(D \cup A) \end{aligned}$$

it follows that $^\sim P(D \cap A) = {}^\sim P(C \cap A)$, whence, and from $^\sim P(C \cap A) \leq {}^\sim P(F \cap A) \leq {}^\sim P(D \cap A)$ we get

$$^\sim P(C/A) = {}^\sim P(F/A) = {}^\sim P(D/A) \tag{4}$$

From (1) and (4) it follows that

$$L(P(C/A)) = L(P(D/A)) = {}^\sim P(F/A) \tag{5}$$

and $L(P(D\mathbf{C}/A))=0$. For the set $F\in\mathfrak{M}(^\sim P)$ we have the following representation:

$$F=C\cup(F\mathbf{C})\ \text{ where }\ C\in\L(V), F\mathbf{C}\subseteq D\mathbf{C}\ \text{ and }\ L(P(D\mathbf{C}/A))=0.$$

So $F\in\mathfrak{M}(L(P(\cdot/A)))$. Since $C\subseteq F\subseteq D$ and $C,D,F\in\mathfrak{M}(L(P(\cdot/A)))$ we have that $L(P(C/A))\le L(P(F/A))\le L(P(D/A))$ which, in view of (5), implies

$$^\sim P(F/A)\le L(P(F/A))\le {}^\sim P(F/A)\qquad\text{i.e.}\qquad L(P(F/A))={}^\sim P(F/A)$$

Later on, whenever $A\in^*\mathfrak{P}(V)$, $P(A)>0$ and $P(A)\approx' 0$, the conditional probability $^\sim P(\cdot/A)$ on $(V,\mathfrak{M}(^\sim P))$ will be denoted by $L(P(\cdot/A))$; assuming that it is a Loeb measure.

We now prove a representation theorem for conditional probability $^\sim P(\cdot/A)$ ($A\in\mathfrak{M}(^\sim P)$ and $^\sim P(A)>0$) on $(V,\mathfrak{M}(^\sim P))$. We shall show that there exists a set $C\in^*\mathfrak{P}(V)$ with $P(C)>0$ and $P(C)\approx' 0$ such that the conditional probability $^\sim P(\cdot/A)$ is equal to the Loeb conditional probability $L(P(\cdot/C))$.

THEOREM 2. *Let $A\in\mathfrak{M}(^\sim P)$ with $^\sim P(A)>0$. Then, there exists a set $C\in^*\mathfrak{P}(V)$ with $P(C)>0$ and $P(C)\approx' 0$ such that*

$$L(P(F/C))={}^\sim P(F/A),\qquad \textit{for any }\ F\in\mathfrak{M}(^\sim P)$$

Proof. According to [**8**] exists a set $C\in^*\mathfrak{P}(V)$ such that $^\sim P(C\triangle A)=0$. We show that $P(C)>0$ and $P(C)\approx' 0$: For sets $A,C\subseteq V$ we have that $C\setminus A\subseteq C\triangle A$, and $A\setminus C\subseteq C\triangle A$, so, by completeness of measure $^\sim P$

$$^\sim P(C\setminus A)={}^\sim P(A\setminus C)=0$$

Since $C=(C\setminus A)\cup(C\cap A)$ and $A=(A\setminus C)\cup(A\cap C)$ and sets A,C satisfy (1)

$$^\sim P(C)={}^\sim P(C\setminus A)+{}^\sim P(C\cap A)={}^\sim P(C\cap A)={}^\sim P(A\setminus C)+(C\cap A)={}^\sim P(A)\quad(2)$$

Hence $P(C)>0$ and $P(C)\approx' 0$,

Let $F\in\mathfrak{M}(^\sim P)$. Then

$$F\cap C=(F\cap A\cap C)\cup((C\setminus A)\cap F)\ \text{ and }\ F\cap A=(F\cap A\cap C)\cup((A\setminus C)\cap F)\quad(3)$$

From (3), $(C\setminus A)\cap F\subseteq A\triangle C$, $(A\setminus C)\cap F\subseteq A\triangle C$, $^\sim P((C\setminus A)\cap F)=0$ and $^\sim P((A\setminus C)\cap F)=0$ it follows that

$$\begin{aligned}^\sim P((A\setminus C)&={}^\sim P(F\cap A\cap C)+{}^\sim P((C\setminus A)\cap F)={}^\sim P(F\cap A\cap C)=\\&={}^\sim P(F\cap A\cap C)+{}^\sim P((A\setminus C)\cap F)={}^\sim P(F\cap A)\end{aligned}\quad(4)$$

Finally, according to theorem 1 (2) and (4) imply

$$L(P(F/C))={}^\sim P(F/C)={}^\sim P(F\cap C)/{}^\sim P(C)={}^\sim P(F\cap A)/{}^\sim P(A)={}^\sim P(F/A)$$

The following theorem is a simple consequence of the Loeb theorem [**8**], but can be quite useful when working in nonstandard probability.

THEOREM 3. *Let* $A \in \mathfrak{M}(^{\sim}P)$ *with* $^{\sim}P(A) > 0$. *Than, for any set* $F \in \mathfrak{M}(^{\sim}P)$ *there exists a set* $C \in^{*} \mathfrak{P}(V)$ *such that* $^{\sim}P(F/A) = {}^{\sim}P(C/A)$.

Proof. For $F \in \mathfrak{M}(^{\sim}P)$, by the Loeb theorem [**8**], there exists a set $C \in^{*} \mathfrak{P}(V)$ such that $^{\sim}P(F \triangle C) = 0$. Since $F \cap A = (F \cap A \cap C) \cup ((F \setminus C) \cap A)$ $C \cap A = (F \cap A \cap C) \cup ((C \setminus F) \cap A)$ $(F \setminus X) \cap A \subseteq F \triangle C$ and $(C \setminus F) \cap A \subseteq FtriangleC$, by the same arguments as in theorem 2, we get that $^{\sim}P(F \cap A) = {}^{\sim}P(C \cap A)$, i.e. $^{\sim}P(F/A) = {}^{\sim}P(C/A)$.

In the second part of this paper we are dealing with internal conditional expectation $E(X/\mathfrak{A})$ of an internal random variable $X : V \to^{*} R$ relative to $\mathfrak{A}$, where $\mathfrak{A}$ is an internal subalgebra of $^{*}\mathfrak{P}(V)$ and $P(A/\mathfrak{A})$ denotes the internal conditional probability of an event $A \in^{*} \mathfrak{P}(V)$ elative to $\mathfrak{A}$.

We consider a hiperfinite probability space $V,^{*} \mathfrak{P}(V), P)$, $A \in^{*} \mathfrak{P}(V)$ and internal subalgebra of $^{*}\mathfrak{P}(V)$. The hyperfinitness of $\mathfrak{A}$ implies, by transfer principle, that $\mathfrak{A}$ is generated by a hyperfinite partition $\{V_1, V_2, \dots V_H\}$ $(H \in^{*} N \setminus N)$ of the set V. It permits us to definer $P(A/\mathfrak{A})$ in the same way as m the case of finite standard probability spaces:

$$P(A/\mathfrak{A})(v) = \sum_{i=1}^{H} P(A/V_i) I_{V_i}(v) \qquad \text{for } v \in V \tag{1}$$

where $P(A/V_i) = P(A \cap V_i)/P(V_i) \qquad i = 1, 2, \dots, H$. Since

$$\begin{aligned} P(A/\mathfrak{A})(v) &= \sum_{i=1}^{H} P(A/V_i) I_{V_i}(v) \\ &= \sum_{i=1}^{H} (P(A \cap V_i)/P(V_i)) I_{V_i}(v) \\ &= \sum_{i=1}^{H} ((P(V_i)^{-1} \sum (P(u) I_A(u), u \in V_i)) I_{V_i}(v) \\ &= \sum_{i=1}^{H} E(I_A/V_i) I_{V_i}(v) \\ &= E(I_A/\mathfrak{A})(v) \qquad \text{i.e.} \end{aligned}$$

$$P(A/\mathfrak{A}) = E(I_A/\mathfrak{A}). \tag{2}$$

in the further work we shall use both (1) and (2) as definitions of internal conditional probability.

For internal random variable $X : V \to^{*} R$ on $(V,^{*} \mathfrak{P}(V), P)$ internal conditional expectation $E(X/\mathfrak{A})$ has already been defined [**6**]. In [**10**] it is proved that $E(X/\mathfrak{A})$ is an S-$\mathfrak{A}$-integrable random variable on $(V,\mathfrak{A},P)$ provided X is S-integrable. This result applied to $P(A/\mathfrak{A})$ implies that $P(A/\mathfrak{A})$ is an S-$\mathfrak{A}$-integrable random variable on $(V,\mathfrak{A},P)$ since I_A is S-integrable, [**9**]. In [**10**] it is proved that

$$E(E(X/\mathfrak{A}))=E(X). \tag{i}$$

Taking (2) as definition of $P(A/\mathfrak{A})$, from (i) it follows, [**9**], that

$$E(P(A/\mathfrak{A}))=P(A) \tag{ii}$$

Results (i) and (ii) make the Theorem of probability completeness for $E(X/\mathfrak{A})$ and $P(A/\mathfrak{A})$ hold for hyperfinite probability spaces.

We now prove a nonstandard version of the well known probability theorem, namely, that conditional probability and expectation relative to σ-subalgebra Ḅ are μ-a. s. unique random variables on $(V,Ḅ,\mu)$, [**11**].

THEOREM 4. *Let $(V,^*\mathfrak{P}(V),P)$ be a hyperfinite probability space, $\mathfrak{A}\subseteq^*\mathfrak{P}(V)$ an internal subalgebra generated by a hyperfinite partition $\{V_1,V_2,\dots,V_H\}$ $(H\in^* N\mathbf{N})$ of V, $X:VV\to^* R$ an S-integrable random variable on $(V,^*\mathfrak{P}(V),P)$ and $A\in^*\mathfrak{P}(V)$. Then*

(i) $\sum(X(v)P(v),v\in U)=\sum(E(X/\mathfrak{A})(v)P(v),v\in U)$ *for* $U\in\mathfrak{A}$

(ii) $E(X/\mathfrak{A})$ *is the P-a. s. unique internal random variable on $(V,\mathfrak{A},P)$ which satisfies* (i)*, i.e. for any other S-integrable $Y:V\to^* R$ on $(V,\mathfrak{A},P)$ satisfying* (i)

$$Y(v)\approx E(X/\mathfrak{A})(v) \qquad P\text{-n. s.}$$

and for any S-integrable $H:V\to^* R$ on $V,\mathfrak{A},P)$ with

$$\sum(|(H(v)-E(X/\mathfrak{A})(v)|P(v),v\in V)\approx 0 \qquad \textit{one has}$$
$$\sum(H(v)P(v),v\in U)\approx\sum(E(X/\mathfrak{A})(v)P(v),v\in U) \qquad \textit{for } U\in\mathfrak{A}.$$

(iii) *For any set $B\in\mathfrak{A}$*

$$P(A\cap B)=\sum(P(A/\mathfrak{A})(v)P(v),v\in B).$$

(iv) $P(A/\mathfrak{A})$ *is the P-a. s. unique internal random variable on $(V,\mathfrak{A},P)$ in the sense given in* (ii).

Proof (i) Since $U=\bigcup_{i=1}^{H}(U\cap V_i)$, for $v\in U\cap V_i E(X/\mathfrak{A})(v)=E(X/U\cap V_i)$

one has

$$\sum(E(X/\mathfrak{A})(v)P(v), v\in U) = \sum_{i=1}^{H}(E(X/\mathfrak{A})(v)P(v), v\in \bigcup_{i=1}^{H}(U\cap V_i))$$
$$= \sum_{i=1}^{H}(E(X/\mathfrak{A})(v)P(v), v\in U\cap V_i))$$
$$= \sum_{i=1}^{H}(E(X/U\cap V_i)P(U\cap V_i)$$
$$= \sum_{i=1}^{H}((P(U\cap V_i)^{-1}\sum(X(u)P(u), u\in U\cap V_i))P(U\cap V_i)$$
$$= \sum(X(u)P(u), u\in \bigcup_{i=1}^{H}(U\cap V_i))$$
$$= \sum(X(u)P(u), u\in U)$$

(ii) Let $F(v) = E(X/\mathfrak{A})(v)$. Then, in view of the Projection Theorem for Integrabllity, [**1**], S-$\mathfrak{A}$-integrability of $F : V \to^* R$ implies that ${}^\sim F : V \to R$ is a ${}^\sim P$-integrable random variable on $(V, \mathfrak{M}(\mathfrak{A}), {}^\sim P)$. If $Y : V \to {}^*R$ is any S-$\mathfrak{A}$-integrable random variable on $(V, \mathfrak{A}, P)$ which satisfies (i) then ${}^\sim Y : V \to R$ is a ${}^\sim P$-integrable random variable on $(V, \mathfrak{M}(\mathfrak{A}), {}^\sim P)$ as well. Therefore, for $U \in \mathfrak{A}$

$$\int_U {}^\sim Y d^\sim P = \mathrm{st}(\sum(Y(u)P(u), u\in U)) = \mathrm{st}(E(F(u)P(u), u\in U)) = \int_U {}^\sim F d^\sim P$$

Let $M \in \mathfrak{M}(\mathfrak{A})$. Then, by [**8**], there exists a set $U \in \mathfrak{A}$ such that ${}^\sim P(U\triangle M) = 0$. Since U satisfies (1), we have

$$\int_M {}^\sim Y d^\sim P = \int_U {}^\sim Y d^\sim P = \int_U {}^\sim F d^\sim P = \int_M {}^\sim F d^\sim P$$

and hence ${}^\sim Y(v) = {}^\sim F(v)$ P-a. s.. This implies that

$$P\{v\in V|\,|Y(v)-F(v)| > n^{-1}\} \approx 0 \quad \text{for every } n\in N.$$

According to [**4**, Robinson's lemma about sequences] there exists $h \in^* N\mathbf{N}$ such that for every $k \in^* N$, $k \le h$

$$P\{v\in V|\,|Y(v)-F(v)| > k^{-1}\} \approx 0$$

Therefore, the set $U = \{v \in V|\,|Y(v)-F(v)| > h^-\} \approx 0$ satisfies: $U \in \mathfrak{A}$, $P(U) \approx 0$, $U \supset \{v \in V|\,Y(v) \neq F(v)\}$ and $Y(v) \approx E(X/\mathfrak{A})(v)$ for $u \notin U$. Hence

$$Y(v) \approx E(X/\mathfrak{A})(v) \qquad P\text{-a. s.}$$

Let $H : V \to^* R$ be an S-$\mathfrak{A}$-integrable random variable on $(V, \mathfrak{A}, P)$ with $\sum(|H(v) - F(v)|P(v), v \in V) \approx 0$. Since ${}^\sim H$, ${}^\sim F$ are ${}^\sim P$-integrable random variables on $(V, \mathfrak{M}(\mathfrak{A}), {}^\sim P)$ and

$$\int\limits_V |{}^\sim H - {}^\sim F| d{}^\sim P = \mathrm{st}(\sum(|H(v) - F(v)|P(v), v \in V)) = 0$$

we have that ${}^\sim H(v) = {}^\sim F(v)$ P-a. s.. Therefore, for $U \in \mathfrak{A}$

$$\sum(H(v)P(v), v \in U) \approx \int\limits_U {}^\sim H d{}^\sim P = \int\limits_U {}^\sim F d{}^\sim P \approx \sum(F(v)P(v), v \in U) \quad \text{i.e.}$$

$$\sum(H(v)P(v), v \in U) \approx \sum(E(X/\mathfrak{A})(v)P(v), v \in U)$$

(iii) According to def (2) for $P(A/\mathfrak{A})$ and (i), for $B \in \mathfrak{A}$ we have

$$\begin{aligned}\sum(P(A/\mathfrak{A})(v)P(v), v \in B) &= \sum(E(I_A/\mathfrak{A})(v)P(v), v \in B)\\ &= \sum(I_A(v)P(v), v \in B) = P(A \cap B)\end{aligned}$$

(iv) Let $F(v) = P(A/\mathfrak{A})(v)$. Then, since F is S-$\mathfrak{A}$-integrable, ${}^\sim F : V \to R$ is a ${}^\sim P$-integrable random variable on $(V, \mathfrak{M}(\mathfrak{A}), {}^\sim P)$, so, for an S-$\mathfrak{A}$-integrable random variable $G : V \to {}^*R$ on $(V, \mathfrak{A}, P)$ which satisfies (iii), we have that for $B \in \mathfrak{A}$

$$\int\limits_B {}^\sim G d{}^\sim P = \mathrm{st}(\sum(G(v)P(v), v \in B)) = \mathrm{st}(\sum(F(v)P(v), v \in B)) = \int\limits_B {}^\sim F d{}^\sim P$$

Hence, by the same arguments as in proof of (ii)

$$G(v) \approx F(v) = P(A/\mathfrak{A})(v) \qquad P\text{-n. s.}$$

For $H : V \to {}^*R$ which is S-$\mathfrak{A}$-integrable and satisfies $\sum(|H(v) - F(v)|P(v), v \in V) \approx 0$, like in (ii), we have that ${}^\sim H(v) = {}^\sim F(v)$ P-a. s. implies that for any $B \in \mathfrak{A}$

$$\begin{aligned}\mathrm{st}(\sum(H(v)P(v), v \in B)) &= \int\limits_B {}^\sim H d{}^\sim P = \int\limits_B {}^\sim F d{}^\sim P\\ &= \mathrm{st}(\sum(F(v)P(v), v \in B)) = \mathrm{st}(P(A \cap B))\end{aligned}$$

i.e.

$$\sum(H(v)P(v), v \in B) \approx P(A \cap B)$$

In [**10**] it is proved that for S-integrable random variable $X : V \to {}^*R$ on $(V, {}^*\mathfrak{P}(V), P)$ the internal conditional expectation $E(X/\mathfrak{A})$ is a lifting of ${}^\sim E({}^\sim X/\mathfrak{M}(\mathfrak{A}))$, ${}^\sim E({}^\sim X/\mathfrak{M}(\mathfrak{A}))$ being the conditional expectation of ${}^\sim X : V \to R$ relative to sub-σ-algebra $\mathfrak{M}(\mathfrak{A}) \subseteq \mathfrak{M}({}^\sim P)$. From this result we derive the following nonstandard characterization of the conditional probability ${}^\sim P(A/\mathfrak{M}(\mathfrak{A}))$, $A \in \mathfrak{M}({}^\sim P)$ on a Loeb space.

THEOREM 5. *Let $(V, {}^*\mathfrak{P}(V), P)$ be a hyperfinite probability space, $\mathfrak{A}$ an internal subalgebra of ${}^*\mathfrak{P}(V)$ and ${}^\sim P(A/\mathfrak{M}(\mathfrak{A}))$ the conditional probability of $A \in \mathfrak{M}({}^\sim P)$ relative to sub-σ-algebra $\mathfrak{M}(\mathfrak{A}) \subseteq \mathfrak{M}({}^\sim P)$. Then there exists a set $B \in {}^*\mathfrak{P}(V)$ such that*

$$\mathrm{st}(P(B/\mathfrak{A})) = {}^\sim P(A/\mathfrak{M}(\mathfrak{A})) \qquad P\text{-a. s.}$$

Proof. For $A \in \mathfrak{M}({}^\sim P)$ there is a set $B \in {}^*\mathfrak{P}(V)$ such that ${}^\sim P(A \triangle B) = 0$. The indicator function I_B

$$I_B(v) = \begin{cases} 1, v \in B \\ 0, v \notin B \end{cases}$$

is an internal, S-integrable random variable on $(V, {}^*\mathfrak{P}(v), P)$. Since

$$P\{v \mid I_A(v) neq I_B(v)\} = P(A \triangle B) = 0$$

I_B is an S-integrable lifting of I_A. In view of [**10**], this implies

$$\begin{gathered} {}^\sim E(I_B/\mathfrak{A}) = {}^\sim E(I_A/\mathfrak{M}(\mathfrak{A})) \qquad P\text{-a. s.} \quad \text{and so} \\ \mathrm{st}(P(B/\mathfrak{A})) = {}^\sim P(A/\mathfrak{M}(\mathfrak{A})) \qquad P\text{-a. s.} \end{gathered}$$

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