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ON THE SPECTRAL RADIUS OF CONNECTED GRAPHS

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Abstract. We prove a general theorem about the maximum spectral radius of connected graphs with n vertices and e edges and use it to determine the graphs with maximum spectral radius when $e \le n+5$ and n is sufficiently large.

1. Introduction. Let $\mathcal{G}(n,e)$ be the set of all graphs with n vertices and e edges in which the vertices are labeled $1, 2, \ldots, n$. Those graphs in $\mathcal{G}(n,e)$ which are connected form a subset which we denote by $\mathcal{H}(n,e)$. The *spectrum of a graph* in $\mathcal{G}(n,e)$ is taken to be the spectrum of its *adjacency matrix* $A = A_G = [a_{ij}]$ which is defined in the usual way as follows. A is a matrix of $0'$s and $1'$s in which $a_{ij} = 1$ if and only if there is an edge joining vertices i and j $(1 \le i, j \le n)$. In particular, A is a symmetric matrix with zero trace. The *spectral radius* $\rho(G)$ of the graph G is defined to be the spectral radius $\rho(A)$ of A, that is the maximum absolute value of an eigenvalue of A. By the Perron-Frobenius theory of nonnegative matrices [**3**], $\rho(A)$ is itself an eigenvalue of A.

In [**1**] Brualdi and Hoffman investigated the maximum spectral radius $g(n,e)$ of a graph in $\mathcal{G}(n,e)$ and showed in particular that for $G = \mathcal{G}(n,e)$, $\rho(G) = g(n,e)$ only if after possibly relabeling the vertices of G, the adjacency matrix $A = [a_{ij}]$ of G satisfies

$$\begin{aligned}(1.1)\qquad &\text{If } 1 \le r < s \le n \text{ and } a_{rs} = 1, \text{ then } a_{kl} = 1 \text{ for all } k \text{ and } 1\\ &\text{with } k < 1,\ 1 \le k \le r, \text{ and } 1 \le l \le s.\end{aligned}$$

Let $\mathcal{G}(n,e)$ denote the subset of $\mathcal{G}(n,e)$ consisting of those graphs whose adjacency matrices $A = [a_{ij}]$ satisfy (1.1), and let $g^*(n,e)$ be the maximum spectral radius of a graph in $\mathcal{G}^*(n,e)$. An example of a graph whose adjacency matrix satisfies (1.1) is given in Figure 0. The result of [**1**] cited above can be restated

$$(1.2)\qquad g(n,e) = g^*(n,e),$$

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and $\rho(G) < g^*(n,e)$ if the vertices of G cannot be labeled so that its adjacency matrix satisfies (1.1).

In this paper we prove the analogue of (1.2) for $\mathcal{H}(n,e)$ and use it to determine the graphs in $\mathcal{H}(n,e)$ with maximum spectral radius when $e \le n+5$. In analogy with the above, we let $\mathcal{H}*(n,e)$ denote the subset of $\mathcal{H}(n,e)$ whose adjacency matrices satisfy (1.1), and we let $h(n,e)$ and $h^*(n,e)$ denote respectively, the maximum spectral radius for graphs in $\mathcal{H}(n,e)$ and $\mathcal{H}^*(n,e)$.

2. The basic theorem. Let $G \in \mathcal{H}^*(n,e)$. Since G is connected there is an edge joining vertex n and some vertex r with $r < n$. Since the adjacency matrix $A = [a_{ij}]$ satisfies (1.1) it follows that $a_{1k} = 1$ for all $k = 2, \ldots, n$ and thus vertex 1 is joined to all other vertices. Note that a graph in $\mathcal{G}(n,e)$ with vertex 1 joined to all other vertices is necessarily connected and thus is in $\mathcal{H}(n,e)$; if, in addition, the graph is in $\mathcal{G}^*(n,e)$, it belongs to $\mathcal{H}^*(n,e)$.

In our proof of the theorem we shall make use of some well known properties of symmetric and nonnegative matrices. These properties will be cited as needed.

THEOREM 2.1. *Let $G \in \mathcal{H}(n,e)$. Then $\rho(G) \le h^*(n,e)$, with equality only if the vertices of G can be labeled so that the resulting graph belongs to $\mathcal{H}^*(n,e)$. In particular $h(n,e) = h^*(n,e)$.*

Proof. Let $G \in \mathcal{H}(n,e) \setminus \mathcal{H}^*(n,e)$, and let $A = [a_{ij}]$ be the adjacency matrix of G with $\rho = \rho(A)$. Since G is connected, A is an irreducible matrix and hence A has a positive eigenvector $x = (x_1, \ldots, x_n)^t$ corresponding to the eigenvalue ρ. We may choose x so that $x^t x = 1$. After possibly relabeling the vertices of G, we may assume that the components of x are monotone nonincreasing. Thus

$$Ax = \rho x, \quad x_1 \ge x_2 \ge \cdots \ge x_n > 0. \tag{2.1}$$

Case 1. $a_{12} = \cdots = a_{1n} = 1$.

Since $G \notin \mathcal{H}^*(n,e)$, there exist integers r and s with $1 < r < s < n$ such that $a_{r,s+1} = 1$ and either $a_{rs} = 0$ or $a_{r-1,s+1} = 0$. Suppose $a_{rs} = 0$. Then we argue as in [**1**]. Let B be the matrix obtained from A by switching the entries a_{rs} and $a_{r,s+1}$ and by switching the entries a_{sr} and $a_{s+1,r}$. Then B is the adjacency matrix of a graph in $\mathcal{H}(n,e)$ (since the non-diagonal entries in its first row are all 1). We calculate that

$$x^t Bx - x^t Ax = 2x_r(x_s - x_{s+1}) \ge 0. \tag{2.2}$$

Suppose equality holds in (2.2) Then $x^t Bx = x^t Ax = \rho$ so that

$$Bx = \rho x = Ax. \tag{2.3}$$

But calculating the s^{th} component of Bx, we see that

$$(Bx)_s = (Ax)_s + x_r > (Ax)_s = \rho x_s.$$

This contradicts (2.3) and hence $x^t Bx > x^t Ax = \rho$. It follows from the maximum characterization of ρ for symmetric matrices [**5**] that $\rho(B) > \rho$. A similar conclusion holds when $a_{r-1,s+1} = 0$. Hence in this case, when $G \notin \mathcal{H}^a st(n,e)$, $\rho(G) < h(n,e)$.

Case 2. $a_{ij} = 0$ for some j with $1 < j \geq n$.

Determine k so that $a_{12} = \cdots = a_{1k} = 1$ and $a_{1,k+1} = 0$. We show how to determine a graph $H \in \mathcal{H}(n,e)$ whose adjacency matrix $B = [b_{ij}]$ satisfies $b_{121} = \cdots = b_{1k} = b_{1,k+1} = 1$ and $\rho(G) < \rho(H)$. Since G is connected, there exists an elementary chain γ which connects vertex 1 to vertex $k+1$. Let p be the first vertex on γ with $p > k$. Let q be the vertex of γ which immediately precedes p. Let G be the graph obtained from G by deleting the edge $[q,p]$ and let H be be obtained from G' by adding the edge $[1,k+1]$. The adjacency matrix $B = [b_{ij}]$ of H satisfies $b_{12} = \cdots = b_{1,k+1} = 1$. We consider two subcases.

Subcase 2.1. $p = k+1$.

Since there is no edge in G joining 1 and $k+1$, it follows that $2 \leq q \leq k$ and hence 1 and q are joined by an edge in G. Thus 1 and q are in the same connected component of G' which implies that H is connected. We calculate that

$$x^tBx - x^tAx = 2x_{k+1}(x_1 - x_q) > 0. \tag{2.4}$$

Supose equality holds in (2.4). Then it follows that (2.3) holds again. But

$$(Bx)_1 = (Ax)_1 + x_{k+1} > (Ax)_1,$$

a contradiction. Thus strict inequality holds in (2.4).

Subcase 2.2. $p > k+1$.

First suppose that $q = 1$. Since p and $k+1$ are joined by a chain in G', p and $k+1$ are in the same connected component of G' and it follows that H is connected. We calculate that

$$x^tBx - x^tAx = 2x_1(x_{k+1} - x_p) \geq 0, \tag{2.5}$$

and as in the above subcase we conclude that strict inequality holds in (2.4).

Now suppose $q > 1$. Since 1 and q are joined by a chain in G', we obtain that H is connected and calculate that

$$x^tBx - x^tAx = 2(x_1x_{k+1} - x_qx_p) = 2x_{k+1}(x_1 - x_q) + 2x_q(x_{k+1} - x_p) \geq 0. \tag{2.6}$$

As above we conclude that strict inequality holds in (2.6).

Thus in this case the matrix B and positive eigenvector x of A satisfy

$$x^tBx > x^tAx = \rho,$$

and we conclude as in Case 1, that $\rho(B) > \rho$. Hence $\rho(G) < h(n,e)$.

Combining cases 1 and 2, we obtain the theorem. $\square$

By the *star* S_n we shall mean the labelled graph in $\mathcal{H}^*(n,n-1)$ drawn in Figure 1. A star with n vertices is any graph isomorphic to S_n.

COROLLARY 2.2. *Let G be a connected graph with n vertices and e edges having the largest possible spectral radius $h(n,e)$. Then G contains a star as a* *spanning tree, and the vertices of G can be labeled so that its adjacency matrix satisfies* (1.1).

COROLLARY 2.3. *Let G in $\mathcal{H}(n,e)$ satisfy $\rho(G) = h(n,e)$. Let $(x_1,\ldots,x_n)^t$ be the positive eigenvector corresponding to the eigenvalue $h(n,e)$ of the adjacency matrix A of G. If r is such that $x_r = \max(x_i : 1 \leq i \leq n)$, then $a_{rj} = 1$ for $j = 1,\ldots,n$ and $j \neq r$.*

In the next section we use Theorem 2.1 to determine the graphs in $\mathcal{H}(n,e)$ which have the largest spectral radius when $e \leq n+5$.

3. Graphs with largest spectral radius. Let G be a tree with n vertices, that is, a graph in $\mathcal{H}(n,e)$ with $e = n-1$. It was shown by Collatz and Singowitz [**2**] and later by Lovász and Pelikán [**4**] that $\rho(G) \leq \sqrt{n-1}$ with equality if and only if G is a star with n vertices. We note here that this result is a special case of Corollary 2.2 which we state as follows.

THEOREM 3.1. *$h(n,n-1) = \sqrt{n-1}$. Moreover, for $G \in \mathcal{H}(n,n-1)$, $\rho(G) = \sqrt{n-1}$, if and only if G is a star with n vertices.*

For later use we observe the following. Let $e \geq n$ and let $G \in \mathcal{H}^*(n,e)$. Then as already observed the adjacency matrix $A = [a_{ij}]$ of G satisfies $a_{12} = \cdots = a_{1n} = 1$, and G contains the star S_n as a spanning subgraph. Since $e \geq n$, it now follows from the theory of nonnegative matrices [**3**] that

$$\rho(G) > \rho(S_n) = \sqrt{n-1}.$$

In our figures to follow all graphs belong to $\mathcal{H}^*(n,e)$ for some e and hence their adjacency matrices satisfy (1.1). The adjacency matrices are used to calculate the characteristic polynomials given.

THEOREM 3.2. *For $e = n$, $n+1$, and $n+2$, the maximum spectral radius $h(n,e)$ of graphs in $\mathcal{H}(n,e)$ occurs uniquely as the spectral radius for those graphs isomorphic to the graphs in Figures 2, 3, and 4, respectively.*

Proof: By Theorem 2.1. a graph in $\mathcal{H}(n,e)$ with maximum spectral radius is isomorphic to a graph in $\mathcal{H}^*(n,e)$. Hence it suffices to determine which graphs in $\mathcal{H}^*(n,e)$ have the largest spectral radius. Recall that a graph in $\mathcal{H}^*(n,e)$ has the star S_n as a spanning subgraph and more generally, its adjacency matrix $A = [a_{ij}]$ satisfies (1.1).

$e = n$: Here $n \geq 3$. The only graph in $\mathcal{H}^*(n,n)$ is the graph in Fig. 2.

$e = n+1$: Here $n \geq 4$. Up to isomorphism there are only two graphs in $\mathcal{H}(n,n+1)$ which have a star as a spanning tree. Only one of these, namely the graph in Fig. 3, belongs to $\mathcal{H}^*(n,n+1)$.

$e = n+2$: Here $n \geq 4$. There are only two graphs in $\mathcal{H}^*(n,n+2)$, namely the graph G_1 in Figure 4 and the graph G_2 in Figure 5 (when $n \geq 5$). The spectral radius $\rho(G_1)$ of G_1 is the maximum root of

$$\varphi_1(\lambda) = \lambda^3 - 2\lambda^2 - (n-1)\lambda + 2(n-4);$$

while $\rho(G_2)$ is the maximum root of $\varphi_2(\lambda) = \lambda^4 - (n+2)\lambda^2 - 6\lambda + 3(n-5)$. We calculate that $\varphi_2(\lambda) - (\lambda+2)\varphi_1(\lambda) = \lambda^2 - (n-1)$, which is positive for $\lambda > \sqrt{n-1}$. Since $\rho(G_2) > \sqrt{n-1}$, it follows that $\rho(G_1) > \rho(G_2)$. $\square$

For $e = n+3$, $n+4$, and $n+5$, we obtain the following characterization of the graph in $\mathcal{H}(n,e)$ with maximum spectral radius valid for n sufficiently large.

THEOREM 3.3. *For $e = n+3$, $n+4$, and $n+5$ and for n sufficiently large, the maximum spectral radius $h(n,e)$ of graphs in $\mathcal{H}(n+e)$ occurs uniquely for those graphs isomorphic to the graph in Figures 6, 7, and 8 respectively.*

Proof. As in the proof of Theorem 3.2. it suffices to determine which graphs in $\mathcal{H}^*(n,e)$ have the largest spectral radius.

$e = n+3$: Here $n \geq 5$. There are exactly two graphs in $\mathcal{H}^*(n, n+3)$, the graph G_1 in Fig. 6 and the graph G_2 in Fig. 9.

The maximum root of $\varphi_1(\lambda) = \lambda^4 - (n+3)\lambda^{-}8\lambda + 4(n-6)$ equals $\rho(G_1)$ while the maximum root of

$$\varphi_2(\lambda) = \lambda^6 - (n+3)\lambda^4 - 10\lambda^3 + (4n-21)\lambda^2 + (2n-8)\lambda - (n-5)$$

equals $\rho(G_2)$. Since $\varphi_1(\lambda)$ has even degree, $\varphi_1(\lambda) > 0$ for negative λ with $|\lambda|$ large. But

$$\varphi_1(-\sqrt{n-2}) = -(n+14) + 8\sqrt{n-2} < 0 \quad \text{for large} \quad n.$$

Hence $\varphi_1(\lambda)$ has a root which is less than $-\sqrt{n-2}$. It follows from Schur's inequality that $\rho(G_1) \leq \sqrt{n+8}$ for n large. Similarly one shows that $\rho(G_2) \leq \sqrt{n+8}$. Hence $\sqrt{n-1} \leq \rho(G_1)$, $\rho(G_2) \leq \sqrt{n+8}$.

Now let $f(\lambda) = \varphi_2(\lambda) - \lambda^2\varphi_1(\lambda) = -2\lambda^3 + 3\lambda^2 + 2(n-4)\lambda - (n-5)$. Then $f(\sqrt{n-1}) = -6\sqrt{n-1} + 2n + 2 > 0$ for large n. Similarly $f(\sqrt{n+8}) > 0$ for large n. Now

$$f'(\lambda) = -6\lambda^2 + 6\lambda + 2(n-4) = 0 \quad \text{when} \quad \lambda = (3 + \sqrt{12n-39})/6.$$

Since $(3 + \sqrt{12n-39})/6 < \sqrt{n-1}$ for n large, it follows that $f'(\lambda) < 0$ for $\lambda \geq \sqrt{n-1}$ and n large. Hence $f(\lambda) > 0$ for $\sqrt{n-1} \leq \lambda \leq \sqrt{n+8}$ when n is large. It now follows that $\rho(G_1) > \rho(G_2)$ for n sufficiently large.

$e = n+4$: Here $n \geq 5$. In this case there are exactly three graphs in $\mathcal{H}^*(n, n+4)$. These are the graph G_3 in Fig. 7 (when $n \geq 7$), the graph G_4 in Fig. 10 (when $n \geq 6$), and the graph G_5 in Fig. 11.

The spectral radii of the graphs G_3, G_4, and G_5 are, respectively, the maximum roots of

$$\varphi_3(\lambda) = \lambda^4 - (n+4)\lambda^2 - 10\lambda + 5(n-7)$$
$$\varphi_4(\lambda) = \lambda^6 - (n+4)\lambda^4 - 12\lambda^3 + (5n-29)\lambda^2 + 2(n-4)\lambda - 2(n-6)$$
$$\varphi_5(\lambda) = \lambda^5 - (n+4)\lambda^3 - 14\lambda^2 + (5n-31)\lambda + 4(n-5).$$

We begin by comparing $\rho(G_3)$ and $\rho(G_4)$. Since $\varphi_3(\lambda)$ has even degree, $\varphi_3(\lambda) > 0$ for negative λ large. But

$$\varphi_3(-\sqrt{n-2}) = -(n+23) + 10\sqrt{n-2} < 0 \text{ for large } n.$$

Hence $\varphi_3(\lambda)$ has a root which is less than $-\sqrt{n-2}$. Schur's inequality implies that $\rho(G_3) \le \sqrt{n+10}$ for n large. Similarly one shows that $\rho(G_4) \le \sqrt{n+10}$ for n large. Thus $\sqrt{n-1} \le \rho(G_3),\ \rho(G_4) \le \sqrt{n+10}$.

Let $f(\lambda) = \varphi_4(\lambda) - \lambda^2\varphi_3(\lambda) = -2\lambda^3 + 6\lambda^2 + 2(n-4)\lambda - 2(n-6)$. We calculate that $f(\sqrt{n-1}) = -6\sqrt{n-1} + 4n + 6 > 0$ for large n. Also $f(\sqrt{n+10}) > 0$ for large n. Now

$$f'(\lambda) = -6\lambda^2 + 12\lambda + 2(n-4) = 0 \quad \text{when} \quad \lambda = (3 + \sqrt{3n-3})/3.$$

Since $(3+\sqrt{3n-3}/3 < \sqrt{n-1}$ for n large, it follows that $f'(\lambda) < 0$ for $\lambda \ge \sqrt{n-1}$ for n large. Hence $f(\lambda) > 0$ for $\sqrt{n-1} \le \lambda \le \sqrt{n+10}$ when n is large. Thus $\rho(G_3) > \rho(G_4)$ for n sufficiently large.

We now compare $\rho(G_3)$ and $\rho(G_5)$. Since $\varphi_5(\lambda)$ has odd degree, $\varphi_5(\lambda) < 0$ for negative λ with $|\lambda|$ large. But

$$\varphi_5(-\sqrt{n-2}) = (n-9)\sqrt{n-2} - 10n + 8 > 0 \ \text{ for large } n.$$

Hence $\varphi_5(\lambda)$ has a root which is less than $-\sqrt{n-2}$. As above we obtain $\rho(G_5) \le \sqrt{n+10}$. Thus $\sqrt{n-1} \le \rho(G_5) \le \sqrt{n+10}$. Let $g(\lambda) = \varphi_5(\lambda) - \lambda\varphi_3(\lambda) = -4(\lambda^2 - \lambda - (n-5))$. Then $g(\lambda) = 0$ when $\lambda = (1 + \sqrt{4n-19})/2$.

Since $(1 + \sqrt{4n-19})/2$ is greater than both $\sqrt{n-1}$ and $\sqrt{n+10}$ for n sufficiently large, it follows that $g(\lambda) > 0$ for $\sqrt{n-1} \le \lambda \le \sqrt{n+10}$ when n is large. Thus $\rho(G_3) > \rho(G_5)$ for n sufficiently large.

$e = n + 5$: We must have $n \ge 5$. There are exactly four graphs in $\mathcal{H}^*(n, n+5)$. These are the graph G_6 in Fig. 8 (when $n \ge 8$), G_7 in Fig. 12 (when $n \ge 6$), G_8 in Fig. 13 (when $n \ge 7$), and G_9 in Fig. 14.

Let

$$\begin{aligned}
\varphi_6(\lambda) &= \lambda^4 - (n+5)\lambda^2 - 12\lambda + 6(n-8)\\
\varphi_7(\lambda) &= \lambda^6 - (n+5)\lambda^4 - 16\lambda^3 + (6n-38)\lambda^2 + 4(n-5)\lambda - 2(n-6)\\
\varphi_8(\lambda) &= \lambda^6 - (n+5)\lambda^4 - 14\lambda^3 + (6n-39)\lambda^2 + 2(n-4)\lambda - 3(n-7)\\
\varphi_9(\lambda) &= \lambda^3 - 3\lambda^2 - (n-1)\lambda + 3(n-5).
\end{aligned}$$

We compare $\rho(G_6)$ with each of $\rho(G_7)$, $\rho(G_8)$, and $\rho(G_9)$.

$\rho(G_6)$ and $\rho(G_7)$: Since $\varphi_6(\lambda)$ has even degree, $\varphi_6(\lambda) > 0$ for negative λ with $|\lambda$ large. But $\varphi_6(-\sqrt{n-2}) = -(n+34) + 12\sqrt{n-2} < 0$ for large n. Hence $\varphi_6(\lambda)$ has a root which is less than $-\sqrt{n-2}$. It then follows from Schur's inequality that $\rho(G_6) \le \sqrt{n+12}$ for n large. Similarly we obtain $\rho(G_7) \le \sqrt{n+12}$. Hence

$$\sqrt{n-1} \le \rho(G_6),\ \ \rho(G_7) \le \sqrt{n+12}.$$

Now let $f(\lambda) = \varphi_7(\lambda) - \lambda^2\varphi_6(\lambda) = -4\lambda^3 + 10\lambda^2 + 4(n-5)\lambda - 2(n-6)$. Then $f(\sqrt{n-1}) = -16\sqrt{n-1} + 8n + 2 > 0$ for large n. Similarly $f(\sqrt{n+12}) > 0$ for large n. Now

$$f'(\lambda) = -12\lambda^2 + 20\lambda + 4(n-5) = 0 \ \text{ when } \ \lambda = (5 + \sqrt{12n-35})/6.$$

Since $(5+\sqrt{12n-35})/6 < \sqrt{n-1}$ for n large, it follows that $f'(\lambda) < 0$ for $\lambda \geq \sqrt{n-1}$ and n large. Hence $f(\lambda) > 0$ for $\sqrt{n-1} \leq \lambda \leq \sqrt{n+12}$ when n is large. It now follows that $\rho(G_6) > \rho(G_7)$ for n sufficiently large.

$\rho(G_6)$ and $\rho(G_8)$: Let

$$g(\lambda) = \varphi_8(\lambda) - \lambda^2\varphi_6(\lambda) = -2\lambda^3 + 9\lambda^2 + (2n-8)\lambda - 3(n-7).$$

Since $\varphi_8(-\sqrt{n-2}) < 0$, we obtain as above that $\sqrt{n-1} \leq \rho(G_8) \leq \sqrt{n+12}$. We calculate that

$$g(\sqrt{n-1}) = -6\sqrt{n-1} + 6n + 12 > 0 \text{ for large } n,$$

and similarly that $g(\sqrt{n+12}) > 0$ for large n. Now

$$g'(\lambda) = -2(3\lambda^2 - 9\lambda - (n-4)) = 0 \text{ when } \lambda = (9+\sqrt{12n+33})/6.$$

Since $(9+\sqrt{12n+33})/6 < \sqrt{n-1}$ for n large, it follows as above that $\rho(G_6) > \rho(G_8)$ for n sufficiently large.

$\rho(G_6)$ and $\rho(G_9)$: We calculate that

$$(\lambda+3)\varphi_9(\lambda) - \varphi_6(\lambda) - 13\lambda^2 + 3n - 3 > 0 \text{ for all } \lambda.$$

Hence $\rho(G_6) > \rho(G_9)$.

This completes the proof of the theorem. $\square$

In the case $e = n+5$, we have verified numerically that the graph in Figure 14 has a larger spectral radius than the graph in Figure 8 for $n \leq 25$. Similarly, in the cases $e = n+3$ and $e = n+4$, for small values of n, the graphs of Figure 6 and 7 do not have the largest spectral radius. Thus the conclusions of Theorem 3.3 do not hold for all n.

We conclude with the following conjecture. Let $e = n+k$ where $k \geq 0$. We have verified that for $k = 0, 1, 3, 4, 5$ and n sufficiently large, there is, up to isomorphism, exactly one graph in $\mathcal{H}(n, n+k)$ with maximum spectral radius and it is the graph obtained from the star S_n by adding the edges from vertex 2 to each of vertices $3, \ldots, k+3$. We *conjecture* that the same conclusions hold for all k with $k \neq 2$.

$$\begin{matrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

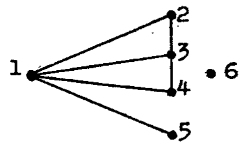

Figure 0. A graph in $\mathcal{G}^*(n, e)$ and its adjacency matrix.

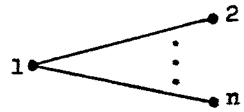

Figure 1. The Star S_n.

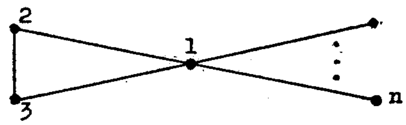

Figure 2. The graph in $\mathcal{H}^*(n, n)$ with n maximum spectral radius. Its characteristic polynomial is $\lambda^{n-4}(\lambda+1)$ $(\lambda^3 - \lambda^2 - (n-1)\lambda + (n-3))$.

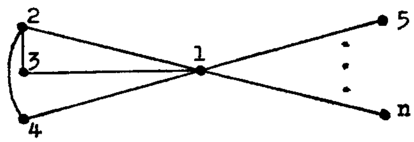

Figure 3. The graph in $\mathcal{H}^*(n, n+1)$ with maximum spectral radius. Its characteristic polynomial is λ^{n-4} $(\lambda^4 - (n+1)\lambda^2 - 4\lambda + 2(n-4))$.

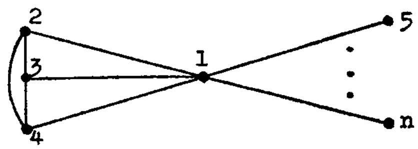

Figure 4. The graph in $\mathcal{H}^*(n, n+2)$ with maximum spectral radius. Its characteristic polynomial is λ^{n-5} $(\lambda+1)^2(\lambda^3 - 2\lambda^2 - (n-1)\lambda + 2(n-4))$.

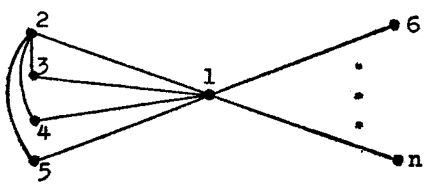

Figure 5. A graph in $\mathcal{H}^*(n, n+2)$. Its characteristic polynomial is λ^{n-4} $(\lambda^4 - (n+2)\lambda^2 - 6\lambda + 3(n-5))$.

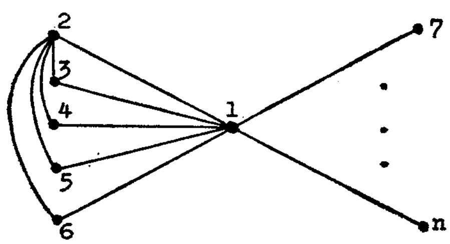

Figure 6. The graph in $\mathcal{H}^*(n, n+3)$ with maximum spectral radius for n sufficiently large. Its characteristic polynomial is $\lambda^{n-2}(\lambda^4 - (n+3)\lambda^2 - 8\lambda + 4(n-6))$.

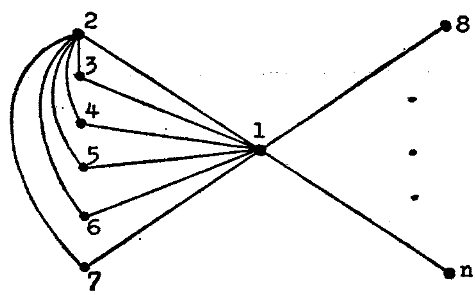

Figure 7. The graph in $\mathcal{H}^*(n, n+4)$ with maximum spectral radius for n sufficiently large. Its characteristic polynomial is $\lambda^{n-4}(\lambda^4 - (n+4)\lambda^2 - 10\lambda + 5(n-7))$.

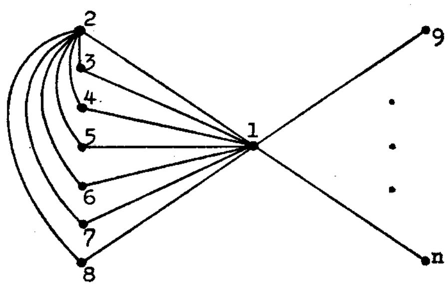

Figure 8. The graph in $\mathcal{H}^*(n, n+5)$ with maximum spectral radius for n sufficiently large. Its characteristic polynomial is $\lambda^{n-4}(\lambda^4 - (n+5)\lambda^2 - 12\lambda + 6(n-8))$.

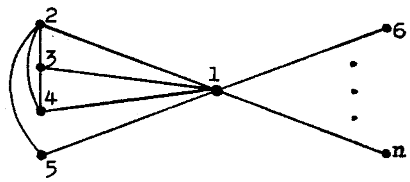

Figure 9. The graph in $\mathcal{H}^*(n, n+3)$. Its characteristic polynomial is $\lambda^{n-6}\ (\lambda^6 - (n+3)\lambda^4 - 10\lambda^3 + (4n - 21)\lambda^2 + 2(n-8))\lambda - (n-5))$.

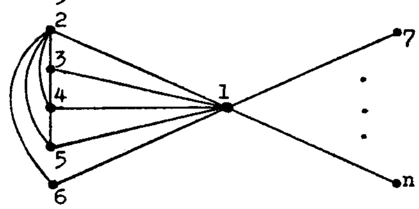

Figure 10. The graph in $\mathcal{H}^*(n, n+4)$. Its characteristic polynomial is $\lambda^{n-6}\ (\lambda^6 - (n+4)\lambda^4 - 12\lambda^3 - (5n - 29)\lambda^2 + 2(n-4)\lambda - 2(n-6))$.

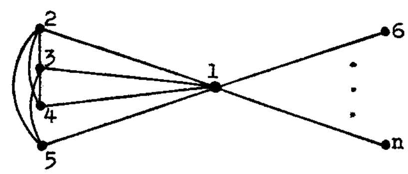

Figure 11. A graph in $\mathcal{H}^*(n, n+4)$. Its characteristic polynomial is $\lambda^{n-5}\ (\lambda^5 - (n+4)\lambda^3 - 14\lambda^2 + (5n - 31)\lambda + (4n-5))$.

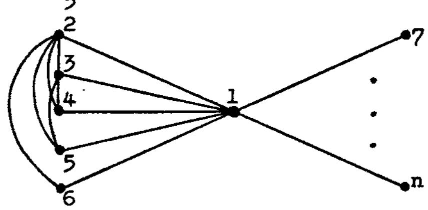

Figure 12. The graph in $\mathcal{H}^*(n, n+3)$ Its characteristic polynomial is $\lambda^{n-6}(\lambda^6 - (n+5)\lambda^4 - 16\lambda^3 + (6n - 38)\lambda^2 + 4(n-5)\lambda - 2(n-6))$.

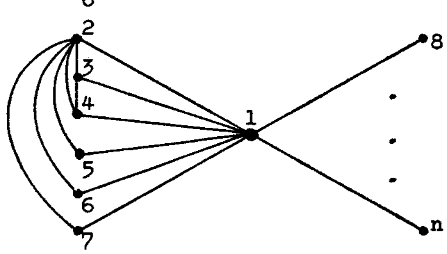

Figure 13. The graph in $\mathcal{H}^*(n, n+4)$ Its characteristic polynomial is $\lambda^{n-6}(\lambda^6 - (n+5)\lambda^4 - 14\lambda^3 + (6n - 39)\lambda^2 + (2n-8)\lambda - 3(n-7))$.

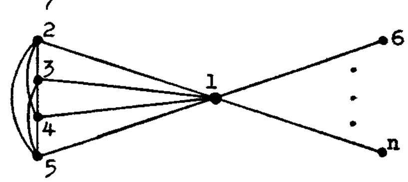

Figure 14. The graph in $\mathcal{H}^*(n, n+5)$ Its characteristic polynomial is $\lambda^{n-6}(\lambda+1)^3(\lambda^3 - 3\lambda^2 - (n-1)\lambda + 3(n-5))$.

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